Case 2 $\lambda_1 = 0$, $\lambda_2 > 0$. Since $\lambda_2 > 0$, (40) yields $\bar{x} = b$. Then (38) yields $f'(b) = \lambda_2$, and since $\lambda_2 > 0$, we obtain the case where $f'(b) > 0$.

Case 3 $\lambda_1 > 0$, $\lambda_2 = 0$. Since $\lambda_1 > 0$, (39) yields $\bar{x} = a$. Then (38) yields the case where $f'(a) = -\lambda_1 < 0$.

Case 4 $\lambda_1 > 0$, $\lambda_2 > 0$. From (39) and (40), we obtain $\bar{x} = a$ and $\bar{x} = b$. This contradiction indicates that Case 4 cannot occur.

Since the constraints are linear, Theorem 11 shows that if $f(x)$ is concave, then (38)–(42) yield the optimal solution to (34). ◆

EXAMPLE 26

A monopolist can purchase up to 17.25 oz of a chemical for \$10/oz. At a cost of \$3/oz, the chemical can be processed into an ounce of product 1, and at a cost of \$5/oz, the chemical can be processed into an ounce of product 2. If x_1 oz of product 1 are produced, product 1 sells for a price of \$30 − x_1 per ounce. If x_2 oz of product 2 are produced, product 2 sells for a price of \$50 − $2x_2$ per ounce. Determine how the monopolist can maximize profits.

Solution Let

$$x_1 = \text{ounces of product 1 produced}$$
$$x_2 = \text{ounces of product 2 produced}$$
$$x_3 = \text{ounces of chemical processed}$$

Then we wish to solve the following NLP:

$$\begin{aligned} \max z &= x_1(30 - x_1) + x_2(50 - 2x_2) - 3x_1 - 5x_2 - 10x_3 \\ \text{s.t.}\quad & x_1 + x_2 \le x_3 \quad \text{or} \quad x_1 + x_2 - x_3 \le 0 \\ & x_3 \le 17.25 \end{aligned} \tag{43}$$

Of course, we should add the constraints $x_1, x_2, x_3 \ge 0$. However, the optimal solution to (43) satisfies the nonnegativity constraints, so the optimal solution to (43) will be optimal for an NLP consisting of (43) with the nonnegativity constraints.

Observe that the objective function in (43) is the sum of concave functions (and is therefore concave), and the constraints in (43) are convex (because they are linear). Thus, Theorem 11 shows that the K-T conditions are necessary and sufficient for (x_1, x_2, x_3) to be an optimal solution to (43). From Theorem 9, the K-T conditions for (43) become

$$30 - 2x_1 - 3 - \lambda_1 = 0 \tag{44}$$
$$50 - 4x_2 - 5 - \lambda_1 = 0 \tag{45}$$
$$-10 + \lambda_1 - \lambda_2 = 0 \tag{46}$$
$$\lambda_1(-x_1 - x_2 + x_3) = 0 \tag{47}$$
$$\lambda_2(17.25 - x_3) = 0 \tag{48}$$
$$\lambda_1 \ge 0 \tag{49}$$
$$\lambda_2 \ge 0 \tag{50}$$

quad1.jpg

As in the previous example, there are four cases to consider:

Case 1 $\lambda_1 = \lambda_2 = 0$. This case cannot occur, because (46) would be violated.

Case 2 $\lambda_1 = 0, \lambda_2 > 0$. If $\lambda_1 = 0$, then (46) implies $\lambda_2 = -10$. This would violate (50).

Case 3 $\lambda_1 > 0, \lambda_2 = 0$. From (46), we obtain $\lambda_1 = 10$. Now (44) yields $x_1 = 8.5$, and (45) yields $x_2 = 8.75$. From (47), we obtain $x_1 + x_2 = x_3$, so $x_3 = 17.25$. Thus, $\bar{x}_1 = 8.5$, $\bar{x}_2 = 8.75$, $\bar{x}_3 = 17.25$, $\bar{\lambda}_1 = 10$, $\bar{\lambda}_2 = 0$ satisfies the K-T conditions.

Case 4 $\lambda_1 > 0, \lambda_2 > 0$. Since Case 3 yields an optimal solution, we need not consider Case 4.

Thus, the optimal solution to (43) is to buy 17.25 oz of the chemical and produce 8.5 oz of product 1 and 8.75 oz of product 2. For Δ small, $\bar{\lambda}_1 = 10$ indicates that if an extra Δ oz of the chemical were obtained at no cost, profits would increase by 10Δ. (Can you see why?) From (46), we find that $\bar{\lambda}_2 = 0$. This implies that the right to purchase an extra Δ oz of the chemical would not increase profits. (Can you see why?) ◆

◆ PROBLEMS

GROUP A

1.[1] A power company faces demands during both peak and off-peak times. If a price of p_1 dollars per kilowatt-hour is charged during the peak time, customers will demand $60 - 0.5p_1$ kwh of power. If a price of p_2 dollars is charged during the off-peak time, customers will demand $40 - p_2$ kwh. The power company must have sufficient capacity to meet demand during both the peak and off-peak times. It costs \$10 per day to maintain each kilowatt-hour of capacity. Determine how the power company can maximize daily revenues less operating costs.

2. Use the K-T conditions to find the optimal solution to the following NLP:

$$\begin{aligned} \max z &= x_1 - x_2 \\ \text{s.t.}\quad x_1^2 + x_2^2 &\le 1 \end{aligned}$$

3. Consider the Giapetto problem of Section 3.1:

$$\begin{aligned} \max z &= 3x_1 + 2x_2 \\ \text{s.t.}\quad 2x_1 + x_2 &\le 100 \\ x_1 + x_2 &\le 80 \\ x_1 &\le 40 \\ x_1 &\ge 0 \\ x_2 &\ge 0 \end{aligned}$$

Find the K-T conditions for this problem and discuss their relation to the dual of the Giapetto LP and the complementary slackness conditions for the LP.

4. If the feasible region for (22) is bounded and contains its boundary points, it can be shown that (22) has an optimal solution. Suppose that the regularity conditions are valid but that the hypotheses of Theorems 11 and 11′ are not valid. If we can prove that only one point satisfies the K-T conditions, why must that point be the optimal solution to the NLP?

5. A total of 160 hours of labor are available each week at \$15/hour. Additional labor can be purchased at \$25/hour. Capital can be purchased in unlimited quantities at a cost of \$5/unit of capital. If K units of capital and L units of labor are available during a week, then $L^{1/2}K^{1/3}$ machines can be produced. Each machine sells for \$270. How can the firm maximize its weekly profits?

6. Use the K-T conditions to find the optimal solution to the following NLP:

$$\begin{aligned} \min z &= (x_1 - 1)^2 + (x_2 - 2)^2 \\ \text{s.t.}\quad -x_1 + x_2 &= 1 \\ x_1 + x_2 &\le 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

7. For Example 26, explain why $\bar{\lambda}_1 = 10$ and $\bar{\lambda}_2 = 0$. (*Hint:* Think about the economic principle that for each product produced, marginal revenue must equal marginal cost.)

[1] Based on Littlechild, "Peak Loads" (1970).

quad2.jpg

7. $\begin{bmatrix} -1 & 0 & 4 & 2 \\ -3 & -1 & 5 & -2 \end{bmatrix}$

8. $\begin{bmatrix} 0 & 3 & -4 & 6 \\ 3 & 1 & -2 & 0 \\ 1 & -2 & -5 & -1 \\ -2 & 5 & 0 & 4 \end{bmatrix}$

9. Solve the game of Example 8.1, which had payoff matrix $\begin{bmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{bmatrix}$. Is this a fair game?

10. Prove Theorem VIII.4.

11. Show that for the payoff matrix of Example 8.17, namely $A = \begin{bmatrix} 2 & 2 \\ -1 & 1 \\ 2 & 2 \end{bmatrix}$, any strategy of the form $[p \quad 0 \quad 1-p]$, where $0 \le p \le 1$, is optimal for player R, and any strategy of the form $[q \quad 1-q]^T$, where $0 \le q \le 1$, is optimal for player C.

8.5 THE SOLUTION OF NONSTRICTLY DETERMINED GAMES

Recall that solving a game amounts to determining optimal strategies for the players of the game. In the preceding section the solution process was easy, since in strictly determined games the players' optimal strategies are pure strategies determined by the saddle point of the payoff matrix. Unfortunately, most games are not strictly determined. In the remainder of this chapter we consider the more difficult problem of solving non-strictly determined games. In the present section we first discuss how the solution process can sometimes be simplified by eliminating certain rows and columns of the game matrix, and then proceed to the solution of 2×2 matrix games. In the final section of this chapter we show how linear programming techniques can be used to solve any $m \times n$ matrix game.

It is sometimes possible to simplify a matrix game by eliminating a number of rows and/or columns from the payoff matrix. Suppose our goal is to determine optimal strategies for players R and C if the game payoff matrix is

$$A = \begin{bmatrix} 6 & 8 & 2 \\ 3 & 5 & 4 \\ 5 & 2 & 1 \end{bmatrix}.$$

We note first that it is never to player R's advantage to select row 3, since every entry in row 1 is greater than the corresponding entry in row 3. Thus we can eliminate row 3 from consideration and reduce matrix A to the new payoff matrix

$$A_1 = \begin{bmatrix} 6 & 8 & 2 \\ 3 & 5 & 4 \end{bmatrix}.$$

It is noteworthy that this reduction of the payoff matrix changes neither the game value nor the players' optimal strategies.

game1.jpg

Similarly, we realize that it is never advantageous for player C to select column 2, since every entry in column 2 is greater than the corresponding entry in either column 1 or column 3. Thus the game matrix can be further reduced to

$$A_2 = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix},$$

still without changing the game value or the players' optimal strategies.

In the vocabulary of game theory, we say that row 1 **dominates** row 3 and that columns 1 and 3 **dominate** column 2. A formal definition appears below.

DEFINITION VIII.5 Let the $m \times n$ matrix $A = [a_{ij}]$ be the payoff matrix for a game. If each entry in the kth row of A is greater than or equal to the corresponding entry in the lth row of A, then the kth row is said to **dominate** the lth row.

If each entry in the kth column of A is less than or equal to the corresponding entry in the lth column of A, then the kth column is said to **dominate** the lth column.

In general, a good starting point for solving a matrix game is to eliminate all dominated rows and columns from the payoff matrix. Since no intelligent player would ever choose to play a dominated row or column, elimination of such a row or column clearly does not affect either player strategy or game value.

In the opening paragraphs of this section we saw that, through elimination of dominated strategies, payoff matrix $\begin{bmatrix} 6 & 8 & 2 \\ 3 & 5 & 4 \\ 5 & 2 & 1 \end{bmatrix}$ could be reduced to $\begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$. Since this 2×2 matrix has no saddle point, we conclude that the game is not strictly determined. Therefore its solution is not immediately obvious. In Examples 8.19 and 8.20 we attack anew our initial problem of determining optimal strategies for players R and C faced with the above payoff matrix.

EXAMPLE 8.19 Solve the game having 2×2 payoff matrix $\begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$.

SOLUTION Solving a matrix game requires that we find optimal strategies for players R and C. Recall that throughout our consideration R's motivation is maximization of gain and C's motivation is minimization of loss.

If the game were to be played only once, R and C would have to exercise their own judgment in deciding what moves to make. R, for instance, might elect to maximize the minimum reward possible on a single play and choose row 2. However, if the game is to be played repeatedly, game theory provides the players with

game2.jpg

optimal strategies which dictate that a player choose one move a given percentage of the time and the other move the remainder of the time.

We first consider this game from player R's point of view. An optimal strategy for R is of the form $\mathbf{P}^* = [p \quad 1-p]$, where p is the probability with which R chooses row 1. Now R's expected winnings are $6p + 3(1-p) = 3p + 3$ if C chooses column 1, and $2p + 4(1-p) = 4 - 2p$ if C chooses column 2. If R can ensure that both

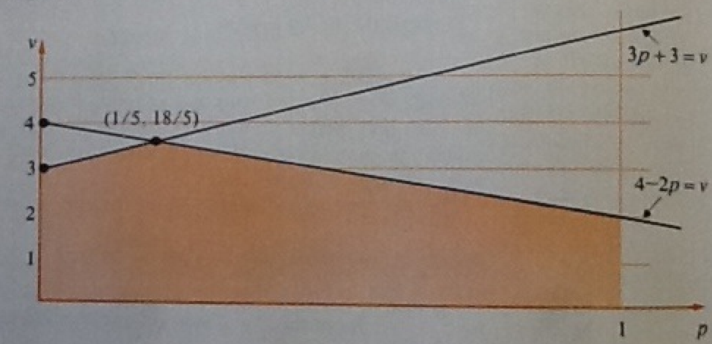
$$3p + 3 \geq v$$

and

$$4 - 2p \geq v,$$

then he is guaranteed a long-run average gain of at least v units, no matter what move player C makes. Note that at this point the value of v is still unknown. Since R's goal is to maximize his gain, he clearly would like the value v to be as large as possible. A graphical representation is helpful at this point. In Figure 8.1 we graph the equations $3p + 3 = v$ and $4 - 2p = v$ and shade the area in which both inequalities are satisfied.

Figure 8.1



A study of Figure 8.1 makes it clear that the largest possible value v is at the intersection of the two lines $v = 3p + 3$ and $v = 4 - 2p$. Solving the equation

$$3p + 3 = 4 - 2p,$$

we find that $p = \frac{1}{5}$, so $v = 3(\frac{1}{5}) + 3 = \frac{18}{5}$. Thus if this game is to be played repeatedly, R's optimal strategy is $\mathbf{P}^* = [\frac{1}{5} \quad \frac{4}{5}]$, which dictates that R play the first row one-fifth of the time and the second row four-fifths of the time.

A similar approach can be used to determine an optimal strategy for player C. Consider $\mathbf{Q}^* = [q \quad 1-q]^T$, where q is the probability with which player C chooses column 1. Since C's goal is to minimize his expected losses, he wants to determine the smallest possible v such that both

his loss if R chooses row 1 $= 6q + 2(1-q) = 4q + 2 \leq v$

and

his loss if R chooses row 2 $= 3q + 4(1-q) = 4 - q \leq v$.

game3.jpg

The solution method is analogous to the one we used to determine R's optimal strategy. Equating

$$4q + 2 = 4 - q,$$

we find that $q = \frac{2}{5}$, and verify that $v = \frac{18}{5}$. Thus C's optimal strategy $\mathbf{Q}^* = [\frac{2}{5} \;\; \frac{3}{5}]^T$ requires that he play column 1 with probability $\frac{2}{5}$ and column 2 with probability $\frac{3}{5}$.

EXAMPLE 8.20 Solve the game having 3×3 payoff matrix

$$\begin{bmatrix} 6 & 8 & 2 \\ 3 & 5 & 4 \\ 5 & 2 & 1 \end{bmatrix}.$$

SOLUTION You may recognize this payoff matrix as the one we considered in the opening paragraphs of this section. At that time we noted that, since row 3 is dominated by row 1 and column 2 is dominated by columns 1 and 3, no intelligent player would choose to play either row 3 or column 2. If row 3 and column 2 are removed, the resulting 2×2 matrix is the one we solved in Example 8.19. It should be clear that the optimal strategies found in Example 8.19 can be expanded to optimal strategies for the present case simply by inserting zero components in the appropriate spots within the strategy vectors. Specifically, R's strategy $[\frac{1}{5} \;\; \frac{4}{5}]$ becomes $[\frac{1}{5} \;\; \frac{4}{5} \;\; 0]$, and C's strategy $[\frac{2}{5} \;\; \frac{3}{5}]^T$ becomes $[\frac{2}{5} \;\; 0 \;\; \frac{3}{5}]^T$.

Theorems VIII.5 and VIII.6, which follow, provide general formulas for the solution of 2×2 matrix games.

THEOREM VIII.5 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the payoff matrix for a 2×2 matrix game. The game is not strictly determined if and only if either (i) both a and d are greater than both b and c, or (ii) both b and c are greater than both a and d.

Proof: Problem 18, Exercise 8.5.

THEOREM VIII.6 Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be the payoff matrix for a 2×2 matrix game that is not strictly determined. Then optimal strategies for R and C are, respectively, $\mathbf{P}^* = [p \;\; 1 - p]$ and $\mathbf{Q}^* = [q \;\; 1 - q]^T$, where

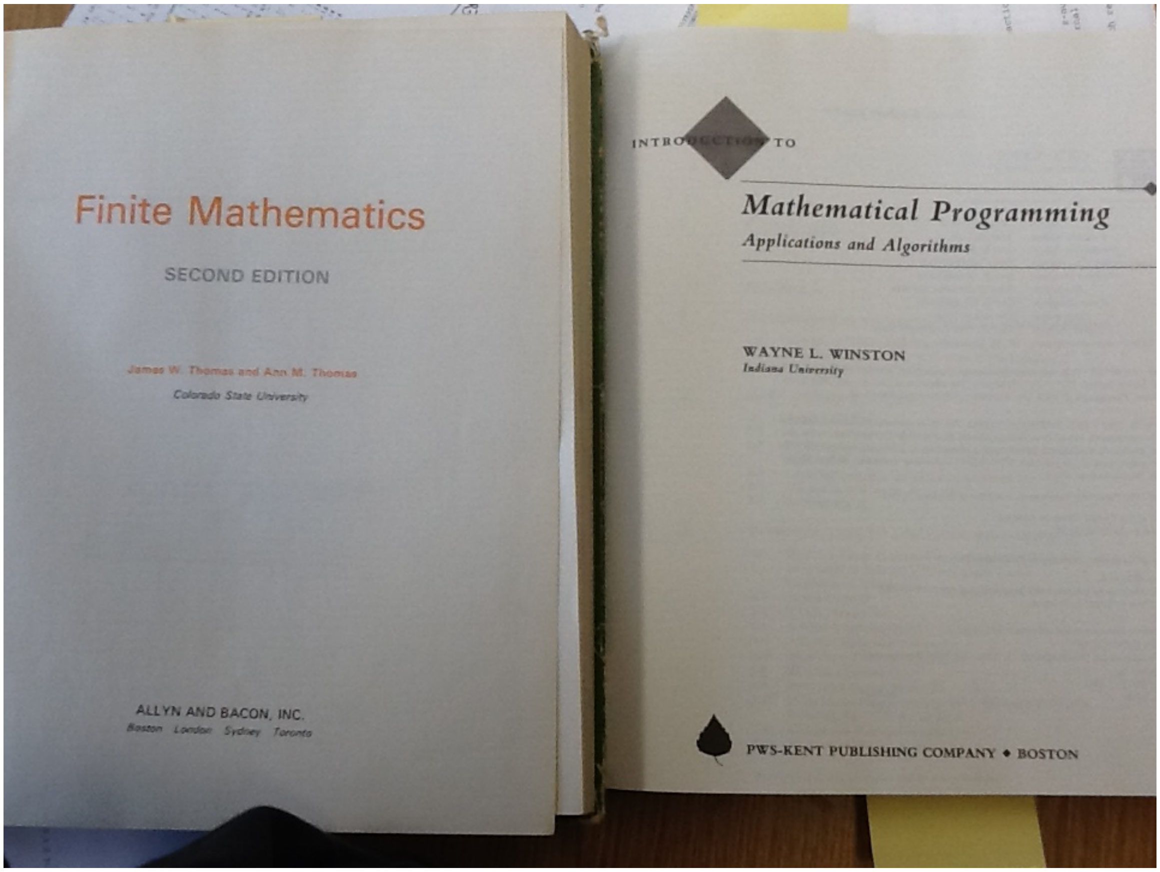
$$p = \frac{d - c}{(a + d) - (b + c)} \quad \text{and} \quad q = \frac{d - b}{(a + d) - (b + c)}.$$

The value of the game is $\dfrac{ad - bc}{(a + d) - (b + c)}$.

Proof: Problem 19, Exercise 8.5.

Note that this theorem applies only if the game is *not* strictly determined. Be sure to verify this hypothesis before applying the theorem.

game4.jpg



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