Feb 22 - 2023 -- notes

$d_k = x_e - x_k \quad (A x_e = b)$

$r_k = b - A x_k = A (x_e - x_k) = A d_k$

$x_{k+1} = x_k + \alpha_k r_k$

$r_{k+1} = r_k - \alpha_k A r_k$

$d_{k+1} = x_e - x_{k+1} = x_e - (x_k + \alpha_k r_k) = d_k - \alpha_k r_k$

$$
\begin{aligned}
(A d_{k+1}, d_{k+1}) &= (A d_{k+1}, d_k - \alpha_k r_k) = (r_{k+1}, d_k - \alpha_k r_k) \\
&= (r_{k+1}, d_k) \\
&= (r_{k+1}, A^{-1} r_k) = (r_k - \alpha_k A r_k , A^{-1} r_k) \\
&= (r_k, A^{-1} r_k) - \alpha_k (r_k , r_k) \\
&= (r_k,A^{-1} r_k)[1 - \alpha_k (r_k , r_k)/ (r_k,A^{-1} r_k)] \\
&= (r_k,A^{-1} r_k)[1 - (r_k,r_k)^2/((Ar_k,r_k) (r_k,A^{-1} r_k))] \\
&\le (r_k,A^{-1} r_k)[1 - 4 \lambda_1\lambda_n/ (\lambda_1+\lambda_n)^2]
\end{aligned}
$$

$(A d_k,d_k)$

====================

convergence of steepest descent:

If $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$ then

$$\frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n} = \frac{\kappa - 1}{\kappa + 1} \qquad \kappa = \text{condition number} = \lambda_1 / \lambda_n$$

====================

MR

$\tilde{x} = x + \alpha r$

want

$\tilde{r} = r - \alpha Ar \perp Ar \quad (r - \alpha Ar, Ar) = 0 \to \alpha = (r,Ar)/(Ar,Ar)$

====================

MR

assumption A is positive definite (not necessarily symmetric)

$(Ax,x) >0$ for all $x \neq 0$ (real)

$(Ax,x) = (x, A^T x) = (A^T x ,x)$

$(Ax,x) = ½ ((A + A^T) x,x)$

$½ (A + A^T)$ = symmetric part of A - it is assumed to be SPD.

CLAIM: there is a $\mu >0$

$(Ax,x) = ½ ((A + A^T) x,x) \ge \mu (x,x)$ for all x $\mu >0$

μ = smallest eigenvalue of symm. part

====================

RN steepest descent

$\tilde{x} = x + \alpha\, d \qquad d = A^T r$
want
$\tilde{r} = r - \alpha\, Ad \perp Ad \qquad (r - \alpha\, Ad, Ad) = 0 \to \alpha = (r,Ad)/(Ad,Ad)$

$\alpha = (r,Ad)/(Ad,Ad) = (A^T r,d)/(Ad,Ad) = (d,d)/(Ad,Ad)$

normal equations:

$A^T A x = A^T b$ (NE) residual $= A^T (b - A x)$

$d = A^T (b - A x)$ = residual for normal equations

steepest descent for NE

$\tilde{x} = x + \alpha\, d$

[$\tilde{d} = d - \alpha\, (A^T A)\, d$]

$\alpha = (d,d) / (A^T A d,d) = = (d,d) / (A d, Ad)$

====================

Krylov methods = residual pol.

$(1 - \alpha_k t)(1 - \alpha_{k-1} t) \; \ldots \; (1 - \alpha_0 t)$

$p_{k+1}(0) = 0 \to p_{k+1}(t) = 1 - t\, q_k(t)$

min degree for which :
$p(A)\, v_1 = 0$ (p of degree k)

GRADE

====================

Arnoldi - step j

$w = A\, v_j$

$w = Av_j - h_{1j} v_1 - h_{2j} v_2 \ldots\ldots - h_{jj} v_j$
$h_{j+1} = \ldots$

$v_{j+1} = w/h_{j+1} \to h_{j+1} v_{j+1} = Av_j - h_{1j} v_1 - h_{2j} v_2 \ldots\ldots - h_{jj} v_j$

$\to h_{1j} v_1 + h_{2j} v_2 \ldots\ldots + h_{jj} v_j + h_{j+1} v_{j+1} = A v_j$

$A v_j = \sum$ (from i=1 to j+1) $h_{ij} v_i$