THE LU FACTORIZATION [2.5]

LU factorization: Motivation

➤ Suppose we have to solve many linear systems

$$Ax = b^{(1)}, \quad Ax = b^{(2)}, \quad \cdots, \quad Ax = b^{(p)}$$

where matrix A is the same - but the right-hand sides are different

➤ Can solve each of them by Gaussian Elimination separately $\longrightarrow$ inefficient

✍ Cost?

➤ Can get the inverse A^{-1} then each solution is of the form $x^{(k)} = A^{-1}b^{(k)}$

✍ Cost? [Using method based on rref seen in Lec. Notes 8]

➤ There is a 3rd option (Best): Exploit "LU factorization of $\boldsymbol{A}$"

➤ Main result is this:

*Gaussian elimination algorithm can provide as a by-product a *factorization* of $\boldsymbol{A}$ into the product of a lower triangular matrix $\boldsymbol{L}$ with ones on the diagonal, and an upper triangular matrix $\boldsymbol{U}$:*

$$\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$$

➤ In addition:

This factorization is obtained at virtually no extra cost.

✍ How would you solve systems with multiple right-hand sides using this? What does this approach cost?

➤ Next: The LU factorization. Where does it come from and how to get it?

LU factorization – Revisiting GE

- We now ignore the right-hand side in GE

Recall: Gaussian elimination amounts to performing $n-1$ successive Gaussian transformations, i.e., multiplications (to the left) by elementary matrices that have ones on the diagonal and the negatives of the multipliers in the column being eliminated.

- Set $A_0 \equiv A$. Then – results of the $n-1$ steps:

$$\begin{aligned} A_1 &= E_1 A_0 \\ A_2 &= E_2 A_1 = E_2 E_1 A_0 \\ A_3 &= E_3 A_2 = E_3 E_2 E_1 A_0 \\ \cdots &= \cdots \\ A_{n-1} &= E_{n-1} E_{n-2} \cdots E_2 E_1 A_0 \end{aligned}$$

- $A_{n-1} \equiv U$ is an upper triangular matrix.
- We have $U = E_{n-1}E_{n-2}\cdots E_2E_1A$ or :

$$A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}}_{L} U \equiv LU$$

- $E_1, E_2, \cdots, E_{n-1}$ are all lower triangular matrices with ones on the diagonal.

✍ What is the inverse of a matrix E_j?

- Each E_j^{-1} is lower triangular with ones on the diagonal.

✍ Show that the product of unit lower triangular matrices is unit lower triangular.

- $L = E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{n-1}^{-1}$ is lower triangular
- L has ones on the diagonal.

- In the end :

$$A = LU \quad \text{with:}$$
$$L = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{n-1}^{-1}$$
$$U = A_{n-1}$$

- Called the LU decomposition (or factorization) of A.

Notes:

- L is Lower triangular, and has ones on the diagonal – We say that it is *unit lower triangular*

- U is the last matrix into which A is transformed from Gaussian elimination. It is *upper triangular.*

- We know how to get U [last matrix in GE]

- The main issue now is: How can we get L?

How do we get L?

- Could we use: $L = E_1^{-1} E_2^{-1} E_3^{-1} \cdots E_{n-1}^{-1}$? Too complex!
- There is a simpler way:

Theorem. Assume that Gaussian elimination can terminate (no division by zero) and let U be the final triangular matrix obtained and L the lower triangular matrix with $l_{ii} = 1$, and, for $i > k$, $l_{ik} = piv_{ik}$, the multiplier used to eliminate row i in step k. Then: $A = LU$.

- $l_{kk} = 1$ and for $i \neq k$, $l_{ik} =$ multiplier a_{ik}/a_{kk} at k-th step of GE.
- The matrix A is the product of a unit lower triangular matrix L and an upper triangular matrix U.

LU factorization - an example

Example: Let $$A = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 5 & 3 \\ 2 & 3 & 9 \end{bmatrix}$$

Step 1 of GE uses the multipliers $l_{21} = -1/2$, $l_{31} = 1/2$.

✍ What is the matrix E_1 in this case?

➤ Resulting matrix: $$A_1 = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 4 & 8 \end{bmatrix}$$

Step 2 of Gaussian Elimination uses the multiplier $l_{32} = 1$.

✍ What is the matrix E_2 ?

- Resulting matrix $A_2 = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} \equiv U$

- Thus: $L = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 1/2 & 1 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 4 & -2 & 2 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix}$

✍ Verify that $A = LU$

✍ LU factorization of the matrix $A = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 5 & 9 \\ 1 & 0 & -12 \end{pmatrix}$

✍ For the same A compute the 3rd column of A^{-1}.

How would you compute the inverse of a matrix given its LU factorization?

Show how to use the LU factorization to solve linear systems with the same matrix $\boldsymbol{A}$ and different right-hand sides $\boldsymbol{b}$.

True or false: "Computing the LU factorization of a matrix $\boldsymbol{A}$ involves more arithmetic operations than solving a linear system $\boldsymbol{Ax} = \boldsymbol{b}$ by Gaussian elimination" ?