LINEAR MAPPINGS [1.8]

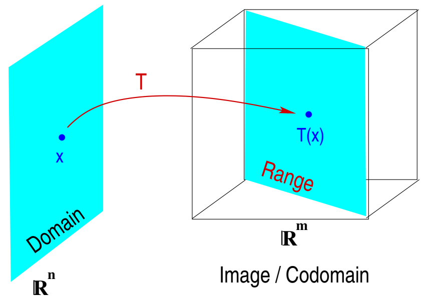
Introduction to linear mappings [1.8]

➤ A transformation or function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to every x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

➤ $\mathbb{R}^n$ is called the domain space of T and $\mathbb{R}^m$ the image space or co-domain of T.

➤ Notation:

$$T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$



➤ $T(x)$ is the image of x under T

Example: Take the mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

Example: Another mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + 5x_2 \end{pmatrix}$$

✍ What is the main difference between these 2 examples?

Definition A mapping T is linear if:

(i) $T(u + v) = T(u) + T(v)$ for u, v in the domain of T

(ii) $T(\alpha u) = \alpha T(u)$ for all $\alpha \in \mathbb{R}$, all u in the domain of T

➤ The mapping of the second example given above is linear - but not for the first one.

➤ If a mapping is linear then $T(0) = 0$. (Why?)

Observation: A mapping is linear if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for all scalars α, β and all u, v in the domain of T.

✍ Prove this

➤ Consequence:

$T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_p u_p) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_p T(u_p)$

➤ Given an $m \times n$ matrix A, consider the special mapping:

$$T: \quad \mathbb{R}^n \longrightarrow \mathbb{R}^m$$
$$x \longrightarrow y = Ax$$

✍ Domain == ??; Image space == ??

➤ From what we saw earlier ['Properties of the matrix-vector product'] such mappings are linear

➤ As it turns out:

If T is linear, there exists a matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$

➤ In plain English: 'A linear mapping can be represented by a matvec'

➤ A is the representation of T.

➤ How can we determine A?

➤ Notation let

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \; j - \text{th row} \qquad x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_j \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix}$$

- Write a vector x in $\mathbb{R}^n$ as $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$.
- Then note that $T(x) = \alpha_1 T(e_1) + \cdots + \alpha_n T(e_n)$
- Therefore the columns of the matrix representation of T must be the vectors $T(e_j)$ for $j = 1, \cdots, n$

✍ Let A be a square matrix. Is the mapping $x \rightarrow x + Ax$ linear? If so find the matrix associated with it.

✍ Same questions for the mapping $x \rightarrow Ax + \alpha x$ - where α is a scalar.

✍ Express the following mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ in matrix/vector form:

$$y_1 = 2x_1 - x_2 + 1$$
$$y_2 = x_2 - x_3 \quad -2$$

➤ Is this a linear mapping?

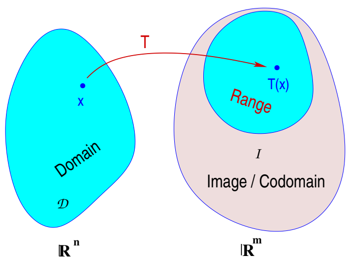
✍ Read Section 1.9 and explore the notions of onto mappings ('surjective') and one-to-one mappings ('injective') in the text. You must at least know the definitions.

✍ A mapping is onto if and only if ....

✍ A mapping is one-to-one if and only if ....

Onto and one-to-one mappings

➤ Let $\boldsymbol{T}$ a mapping – not necessarily linear for now – from a domain set $\boldsymbol{\mathcal{D}}$ (subset of $\mathbb{R}^n$) into an image set $\boldsymbol{\mathcal{I}}$ (subset of $\mathbb{R}^m$)



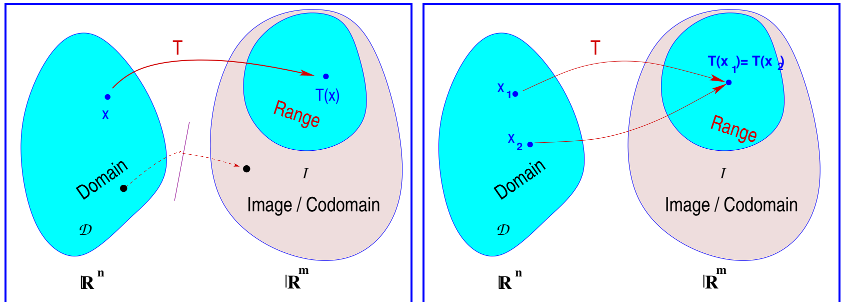
➤ The range of $\boldsymbol{T}$ is the set of all possible vectors of the form $\boldsymbol{T(x)}$ for $\boldsymbol{x} \in \boldsymbol{\mathcal{D}}$.

➤ We say that $\boldsymbol{T}$ is onto if *for every* $\boldsymbol{y}$ *in* $\boldsymbol{\mathcal{I}}$ *there is at least one* $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{D}}$ *such that* $\boldsymbol{y} = \boldsymbol{T(x)}$.

➤ In other words $\boldsymbol{T}$ is onto if the range of $\boldsymbol{T}$ equals all of $\boldsymbol{\mathcal{I}}$

➤ We say that $\boldsymbol{T}$ is one-to-one if *for every* $\boldsymbol{y}$ *in* $\boldsymbol{\mathcal{I}}$ *there is at most one* $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{D}}$ *such that* $\boldsymbol{y} = \boldsymbol{T(x)}$.

➤ In other words if $\boldsymbol{T(u_1) = T(u_2)}$ then we must have $\boldsymbol{u_1 = u_2}$



Not onto — Not one-to-one

➤ Now consider linear mappings: let $\boldsymbol{T}$ represented by a matrix $\boldsymbol{A}$

➤ Now: Domain $\boldsymbol{\mathcal{D}}$ is all of $\mathbb{R}^n$ and Image set $\boldsymbol{\mathcal{I}}$ is all of $\mathbb{R}^m$.

➤ So: $\boldsymbol{A}$ is one-to-one when every $\boldsymbol{y}$ in $\mathbb{R}^m$ is 'reached' by $\boldsymbol{A}$, i.e., if every $\boldsymbol{y}$ in $\mathbb{R}^m$ can be written as $\boldsymbol{y} = \boldsymbol{Ax}$ for some $\boldsymbol{x} \in \mathbb{R}^n$. Since $\boldsymbol{Ax}$ is a linear combination of the columns of $\boldsymbol{A}$, this means that:

➤ $\boldsymbol{A}$ *is onto iff the span of the columns of* $\boldsymbol{A}$ *equals* $\mathbb{R}^m$

✍ Show that $\boldsymbol{A}$ *is one-to-one iff the columns of* $\boldsymbol{A}$ *are linearly independent.*

✍ Find a $\boldsymbol{3 \times 3}$ example of a mapping that is not onto

✍ Finf a $\boldsymbol{3 \times 3}$ example of a mapping that is not one-to-one.

MATRIX OPERATIONS [2.1]

Matrix operations

➤ If A is an $m \times n$ matrix (m rows and n columns) –then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A.

Column j

$$\text{Row } i \rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots a_{in} \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots a_{mn} \end{bmatrix} = A$$

$$\uparrow \quad \uparrow \quad \uparrow$$
$$a_1 \quad a_j \quad a_n$$

➤ The number a_{ij} is the ith entry (from the top) of the jth column

➤ Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$ called a column vector

➤ The columns are denoted by $a_1, ..., a_n$, and the matrix A is written as $A = [a_1, a_2, \cdots, a_n]$

➤ The diagonal entries in an $m \times n$ matrix A are a_{11}, a_{22}, a_{33}, ..., and they form the main diagonal of A.

➤ A diagonal matrix is a matrix whose nondiagonal entries are zero

➤ An important example is the $n \times n$ identity matrix, I_n (each diagonal entry equals one) - Example:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

➤ Another important matrix is the zero matrix (all entries are 0). It is denoted by O.

Equality of two matrices: Two matrices A and B are equal if they have the same size (they are both $m \times n$) and if their entries are all the same.

$$a_{ij} = b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Sum of two matrices: If A and B are $m \times n$ matrices, then their sum $A + B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries in A and B.

➤ If we call C this sum we can write:

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

$$\begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix} = ??; \qquad \begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix} = ??$$

scalar multiple of a matrix If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

$$(\alpha A)_{ij} = \alpha a_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Theorem Let A, B, and C be matrices of the same size, and let α and β be scalars. Then

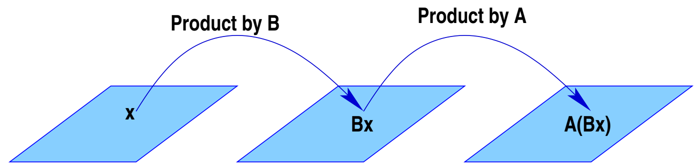
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $\alpha(A + B) = \alpha A + \alpha B$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = (\alpha\beta)A$

✍ Prove all of the above equalities

Matrix Multiplication

➤ When a matrix B multiplies a vector x, it transforms x into the vector Bx.

➤ If this vector is then multiplied in turn by a matrix A, the resulting vector is $A(Bx)$.



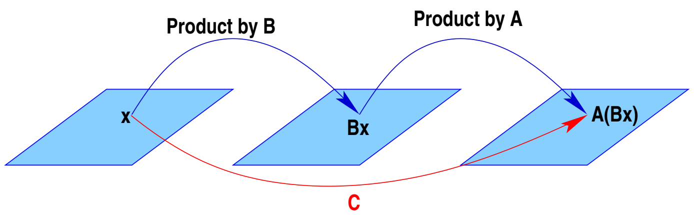
➤ Thus $A(Bx)$ is produced from x by a composition of mappings– the linear transformations induced by B and A.

➤ Note: $x \rightarrow yA(Bx)$ is a linear mapping (prove this).

Goal: to represent this composite mapping as a multiplication by a single matrix, call it C for now, so that

$$A(Bx) = Cx$$

.



➤ Assume A is $m \times n$, B is $n \times p$, and x is in $\mathbb{R}^p$

➤ Denote the columns of B by $b_1, \cdots, b_p$ and the entries in x by $x_1, \cdots, x_p$. Then:

$$Bx = x_1 b_1 + \cdots + x_p b_p$$

➤ By the linearity of multiplication by A:

$$A(Bx) = A(x_1 b_1) + \cdots + A(x_p b_p) = x_1 A b_1 + \cdots + x_p A b_p$$

➤ The vector $A(Bx)$ is a linear combination of $Ab_1, \cdots, Ab_p$, using the entries in x as weights.

➤ In matrix notation, this linear combination is written as

$$A(Bx) = [Ab_1, Ab_2, \cdots Ab_p].x$$

➤ Thus, multiplication by $[Ab_1, Ab_2, \cdots, Ab_p]$ transforms x into $A(Bx)$.

➤ Therefore the desired matrix C is the matrix

$$C = [Ab_1, Ab_2, \cdots, Ab_p]$$

➤ Denoted by AB

Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the matrix whose p columns are $Ab_1, \cdots, Ab_p$. That is:

$$AB = A[b_1, b_2, \cdots, b_p] = [Ab_1, Ab_2, \cdots, Ab_p]$$

➤ Important to remember that :

Multiplication of matrices corresponds to composition of linear transformations.

✍ Operation count: How many operations are required to perform product AB?

✍ Compute AB when

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

✍ Compute AB when

$$A = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 3 & -2 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -2 & 2 \\ 2 & 1 & -2 \\ -1 & 3 & 2 \end{bmatrix}$$

✍ Can you compute AB when

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ -1 & 4 \end{bmatrix} ?$$

Row-wise matrix product

➤ Recall what we did with matrix-vector product to compute a single entry of the vector Ax

➤ Can we do the same thing here? i.e., How can we compute the entry c_{ij} of the product AB without computing entire columns?

✍ Do this to compute entry $(2, 2)$ in the first example above.

✍ Operation counts: Is more or less expensive to perform the matrix-vector product row-wise instead of column-wise?

Properties of matrix multiplication

Theorem Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B + C) = AB + AC$ (left distributive law)
- $(B + C)A = BA + CA$ (right distributive law)
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$ for any scalar α
- $I_m A = AI_n = A$ (product with identity)

✍ If $AB = AC$ then $B = C$ ('simplification') : True-False?

✍ If $AB = 0$ then either $A = 0$ or $B = 0$: True or False?

✍ $AB = BA$: True or false??

Square matrices. Matrix powers

- Important particular case when $n = m$ - so matrix is $n \times n$
- In this case if x is in $\mathbb{R}^n$ then $y = Ax$ is also in $\mathbb{R}^n$
- AA is also a square $n \times n$ matrix and will be denoted by A^2
- More generally, the matrix A^k is the matrix which is the product of k copies of A:

$$A^1 = A; \quad A^2 = AA; \quad \cdots \quad A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

- For consistency define A^0 to be the identity: $A^0 = I_n$,
- $A^l \times A^k = A^{l+k}$ – Also true when k or l is zero.

Transpose of a matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem : Let A and B denote matrices whose sizes are appropriate for the following sums and products.
- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(\alpha A)^T = \alpha A^T$ for any scalar α
- $(AB)^T = B^T A^T$

More on matrix produts

- Recall: Product of the matrix A by the vector x:

$$\overset{y}{\begin{bmatrix}\beta_1\\ \vdots \\ \beta_j \\ \vdots \\ \beta_n\end{bmatrix}} = \overset{A}{\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots a_{in} \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots a_{mn}\end{bmatrix}} \overset{x}{\begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n\end{bmatrix}}$$

$$= \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$$

- x, y are vectors; y is the result of $A \times x$.
- $a_1, a_2, \ldots, a_n$ are the columns of A

- $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the components of x [scalars]
- $\alpha_1 a_1$ is the first column of A multiplied by the scalar α_1 which is the first component of x.
- $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$ is a linear combination of $a_1, a_2, \cdots, a_n$ with weights $\alpha_1, \alpha_2, \ldots, \alpha_n$.
- This is the 'column-wise' form of the 'matvec'

Example: $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ $x = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$ $y = ?$

- Result:

$$y = -2 \times \begin{bmatrix}1\\0\end{bmatrix} + 1 \times \begin{bmatrix}2\\-1\end{bmatrix} - 3 \times \begin{bmatrix}-1\\3\end{bmatrix} = \begin{bmatrix}3\\-10\end{bmatrix}$$

- Can get i-th component of the result y without the others:

$$\beta_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \cdots + \alpha_n a_{in}$$

Example: In the above example extract β_2

$$\beta_2 = (-2) \times 0 + (1) \times (-1) + (-3) \times (3) = -10$$

- Can compute $\beta_1, \beta_2, \cdots, \beta_m$ in this way.
- This is the 'row-wise' form of the 'matvec'

Matrix-Matrix product

- When A is $m \times n$, B is $n \times p$, the product AB of the matrices A and B is the $m \times p$ matrix defined as

$$AB = [Ab_1, Ab_2, \cdots, Ab_p]$$

- Each Ab_j is a matrix-vector product: the product of A by the j-th column of B. Matrix AB has dimension $m \times p$
- Can use what we know on matvecs to perform the product

1. Column form – In words: *"The j-th column of AB is a linear combination of the columns of A, with weights $b_{1j}, b_{2j}, \cdots, b_{nj}$"* (entries of j-th col. of B)

Example: $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ -3 & 2 \end{bmatrix}$ $AB = ?$

Result:
$$B = \left[\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right] = \begin{bmatrix} 3 & -6 \\ -10 & 8 \end{bmatrix}$$

- First column has been computed before: it is equal to: (-2)*(col. 1 of A) + (1)*(col. 2 of A) + (-3)*(col. 3 of A)
- Second column is equal to: (1)*(col. 1 of A) + (-2)*(col. 2 of A) + (2)*(col. 3 of A)

2. If we call C the matrix $C = AB$ what is c_{ij}? From above:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} + \cdots + a_{in}b_{nj}$$

- Fix j and run i $\longrightarrow$ column-wise form just seen

3. Fix i and run j $\longrightarrow$ row-wise form

Example: Get second row of AB in previous example.

$$c_{2j} = a_{21}b_{1j} + a_{22}b_{2j} + a_{23}b_{3j}, \quad j = 1, 2$$

- Can be read as : $\boxed{c_{2:} = a_{21}b_{1:} + a_{22}b_{2:} + a_{23}b_{3:}}$, or in words:

row2 of C = a_{21} (row1 of B) + a_{22} (row2 of B) + a_{23} (row3 of B)
= 0 (row1 of B) + (-1) (row2 of B) + (3) (row3 of B)
= $[-10 \quad 8]$