LINEAR MAPPINGS [1.8]

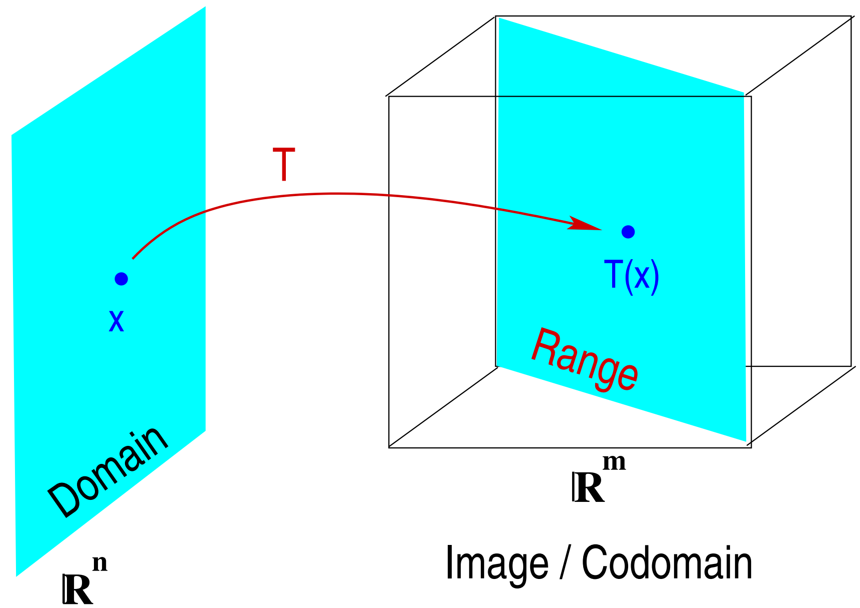
Introduction to linear mappings [1.8]

➤ A transformation or function or mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule which assigns to every $\boldsymbol{x}$ in $\mathbb{R}^n$ a vector $\boldsymbol{T}(\boldsymbol{x})$ in $\mathbb{R}^m$.

➤ $\mathbb{R}^n$ is called the domain space of $\boldsymbol{T}$ and $\mathbb{R}^m$ the image space or co-domain of $\boldsymbol{T}$.

➤ Notation:

$$\boldsymbol{T} : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$



➤ $\boldsymbol{T}(\boldsymbol{x})$ is the image of $\boldsymbol{x}$ under $\boldsymbol{T}$

Example: Take the mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T: \quad \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 x_2 \\ x_1^2 + x_2^2 \end{pmatrix}$$

Example: Another mapping from $\mathbb{R}^2$ to $\mathbb{R}^3$:

$$T: \quad \mathbb{R}^2 \quad \longrightarrow \quad \mathbb{R}^3$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longrightarrow T(x) = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \\ x_1 + 5x_2 \end{pmatrix}$$

✍ What is the main difference between these 2 examples?

Definition A mapping T is linear if:

(i) $T(u+v) = T(u) + T(v)$ for u, v in the domain of T

(ii) $T(\alpha u) = \alpha T(u)$ for all $\alpha \in \mathbb{R}$, all u in the domain of T

- The mapping of the second example given above is linear - but not for the first one.

- If a mapping is linear then $T(0) = 0$. (Why?)

Observation: A mapping is linear if and only if

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

for all scalars α, β and all u, v in the domain of T.

✍ Prove this

- Consequence:

$$T(\alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_p u_p) = \alpha_1 T(u_1) + \alpha_2 T(u_2) + \cdots + \alpha_p T(u_p)$$

➤ Given an $m \times n$ matrix A, consider the special mapping:

$$\begin{aligned} T: \quad \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ x \quad &\longrightarrow y = Ax \end{aligned}$$

✍ Domain == ??; Image space == ??

➤ From what we saw earlier ['Properties of the matrix-vector product'] such mappings are linear

➤ As it turns out:

If T is linear, there exists a matrix A such that $T(x) = Ax$ for all x in $\mathbb{R}^n$

➤ In plain English: 'A linear mapping can be represented by a matvec'

➤ A is the representation of T.

- How can we determine A?
- Notation let

$$e_j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} j - \text{th row} \qquad x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_j \\ \vdots \\ \vdots \\ \alpha_n \end{bmatrix}$$

- Write a vector x in $\mathbb{R}^n$ as $x = \alpha_1 e_1 + \cdots + \alpha_n e_n$.
- Then note that $T(x) = \alpha_1 T(e_1) + \cdots + \alpha_n T(e_n)$
- Therefore the columns of the matrix representation of T must be the vectors $T(e_j)$ for $j = 1, \cdots, n$

✍ Let A be a square matrix. Is the mapping $x \rightarrow x + Ax$ linear? If so find the matrix associated with it.

✍ Same questions for the mapping $x \rightarrow Ax + \alpha x$ - where α is a scalar.

✍ Express the following mapping from $\mathbb{R}^3$ to $\mathbb{R}^2$ in matrix/vector form:

$$\begin{aligned} y_1 &= 2x_1 - x_2 + 1 \\ y_2 &= x_2 - x_3 \quad -2 \end{aligned}$$

➤ Is this a linear mapping?

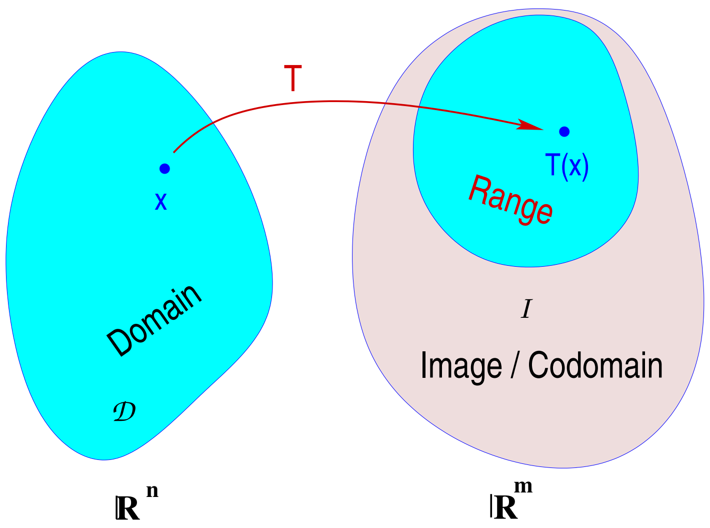
✍ Read Section 1.9 and explore the notions of onto mappings ('surjective') and one-to-one mappings ('injective') in the text. You must at least know the definitions.

✍ A mapping is onto if and only if ....

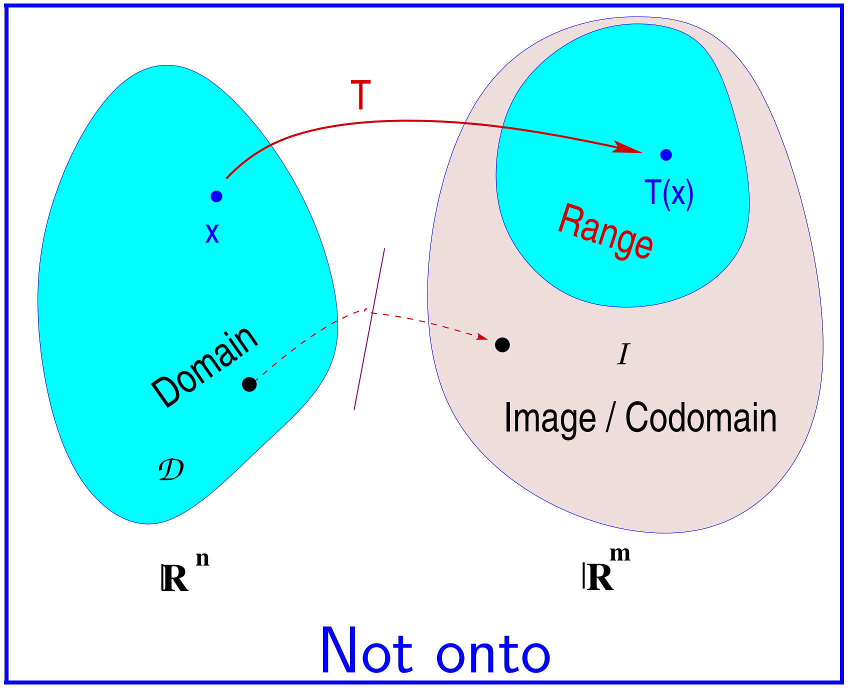
✍ A mapping is one-to-one if and only if ....

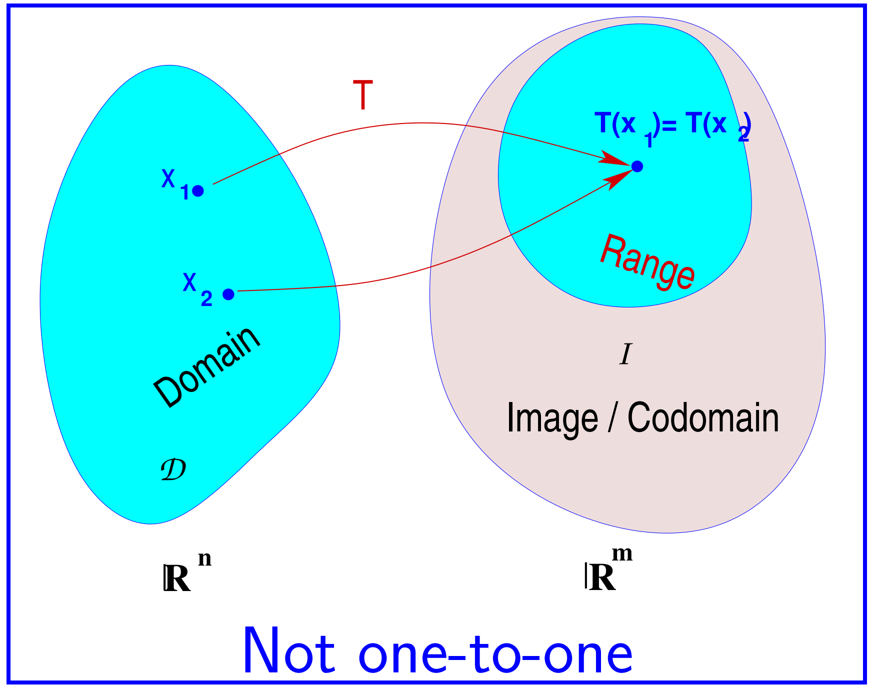
Onto and one-to-one mappings

- Let $\boldsymbol{T}$ a mapping – not necessarily linear for now – from a domain set $\boldsymbol{\mathcal{D}}$ (subset of $\mathbb{R}^n$) into an image set $\boldsymbol{\mathcal{I}}$ (subset of $\mathbb{R}^m$)
- The range of $\boldsymbol{T}$ is the set of all possible vectors of the form $\boldsymbol{T}(\boldsymbol{x})$ for $\boldsymbol{x} \in \boldsymbol{\mathcal{D}}$.



- We say that $\boldsymbol{T}$ is onto if *for every* $\boldsymbol{y}$ *in* $\boldsymbol{\mathcal{I}}$ *there is at least one* $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{D}}$ *such that* $\boldsymbol{y} = \boldsymbol{T}(\boldsymbol{x})$.
- In other words $\boldsymbol{T}$ is onto if the range of $\boldsymbol{T}$ equals all of $\boldsymbol{\mathcal{I}}$
- We say that $\boldsymbol{T}$ is one-to-one if *for every* $\boldsymbol{y}$ *in* $\boldsymbol{\mathcal{I}}$ *there is at most one* $\boldsymbol{x}$ *in* $\boldsymbol{\mathcal{D}}$ *such that* $\boldsymbol{y} = \boldsymbol{T}(\boldsymbol{x})$.
- In other words if $\boldsymbol{T}(\boldsymbol{u}_1) = \boldsymbol{T}(\boldsymbol{u}_2)$ then we must have $\boldsymbol{u}_1 = \boldsymbol{u}_2$





- Now consider linear mappings: let T represented by a matrix A
- Now: Domain $\mathcal{D}$ is all of $\mathbb{R}^n$ and Image set $\mathcal{I}$ is all of $\mathbb{R}^m$.
- So: A is one-to-one when every y in $\mathbb{R}^m$ is 'reached' by A, i.e., if every y in $\mathbb{R}^m$ can be written as $y = Ax$ for some $x \in \mathbb{R}^n$. Since Ax is a linear combination of the columns of A, this means that:
- A *is onto iff the span of the columns of* A *equals* $\mathbb{R}^m$

✍ Show that $\boldsymbol{A}$ *is one-to-one iff the columns of* $\boldsymbol{A}$ *are linearly independent.*

✍ Find a $\mathbf{3} \times \mathbf{3}$ example of a mapping that is not onto

✍ Finf a $\mathbf{3} \times \mathbf{3}$ example of a mapping that is not one-to-one.

MATRIX OPERATIONS [2.1]

Matrix operations

➤ If A is an $m \times n$ matrix (m rows and n columns) –then the scalar entry in the ith row and jth column of A is denoted by a_{ij} and is called the (i, j)-entry of A.

$$
\begin{array}{c}
\text{Column } j \\
\downarrow
\end{array}
$$

$$
\text{Row } i \rightarrow \begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & \boxed{a_{ij}} & \cdots a_{in} \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots a_{mn} \end{bmatrix} = A
$$

$$
\begin{array}{ccc}
\uparrow & \uparrow & \uparrow \\
a_1 & a_j & a_n
\end{array}
$$

- The number a_{ij} is the ith entry (from the top) of the jth column
- Each column of A is a list of m real numbers, which identifies a vector in $\mathbb{R}^m$ called a column vector
- The columns are denoted by $a_1, ..., a_n$, and the matrix A is written as $A = [a_1, a_2, \cdots, a_n]$

- The diagonal entries in an $m \times n$ matrix A are a_{11}, a_{22}, a_{33}, ..., and they form the main diagonal of A.

- A diagonal matrix is a matrix whose nondiagonal entries are zero

- An important example is the $n \times n$ identity matrix, I_n (each diagonal entry equals one) - Example:

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Another important matrix is the zero matrix (all entries are 0). It is denoted by O.

Equality of two matrices: Two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are equal if they have the same size (they are both $\boldsymbol{m} \times \boldsymbol{n}$) and if their entries are all the same.

$$a_{ij} = b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Sum of two matrices: If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $\boldsymbol{m} \times \boldsymbol{n}$ matrices, then their sum $\boldsymbol{A} + \boldsymbol{B}$ is the $\boldsymbol{m} \times \boldsymbol{n}$ matrix whose entries are the sums of the corresponding entries in $\boldsymbol{A}$ and $\boldsymbol{B}$.

➤ If we call $\boldsymbol{C}$ this sum we can write:

$$c_{ij} = a_{ij} + b_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

✍

$$\begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix} = ??; \qquad \begin{bmatrix} 4 & 0 & 5 \\ 1 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -3 \\ 2 & -2 \end{bmatrix} = ??$$

scalar multiple of a matrix If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

$$(\alpha A)_{ij} = \alpha a_{ij} \quad \text{for all } i = 1, \cdots, m, \quad j = 1, \cdots, n$$

Theorem Let A, B, and C be matrices of the same size, and let α and β be scalars. Then

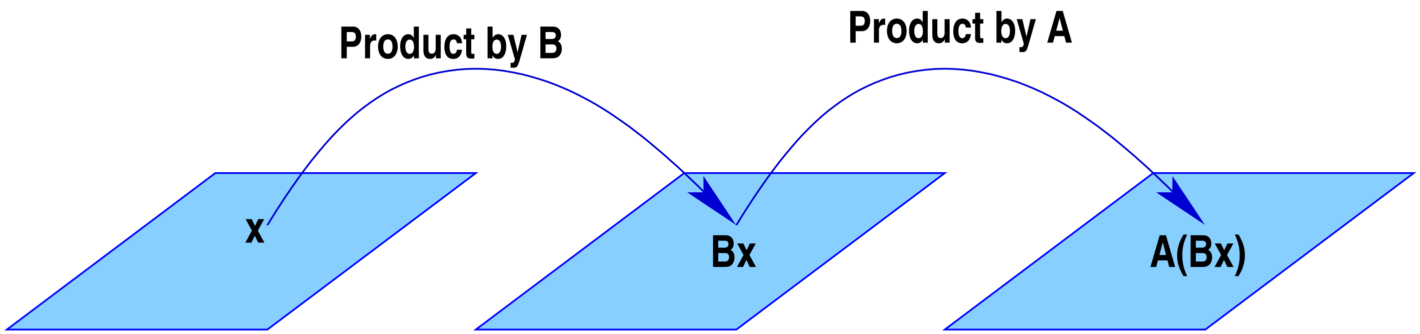
- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + 0 = A$
- $\alpha(A + B) = \alpha A + \alpha B$
- $(\alpha + \beta)A = \alpha A + \beta A$
- $\alpha(\beta A) = (\alpha\beta)A$

✍ Prove all of the above equalities

Matrix Multiplication

- When a matrix $\boldsymbol{B}$ multiplies a vector $\boldsymbol{x}$, it transforms $\boldsymbol{x}$ into the vector $\boldsymbol{B}\boldsymbol{x}$.

- If this vector is then multiplied in turn by a matrix $\boldsymbol{A}$, the resulting vector is $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$.



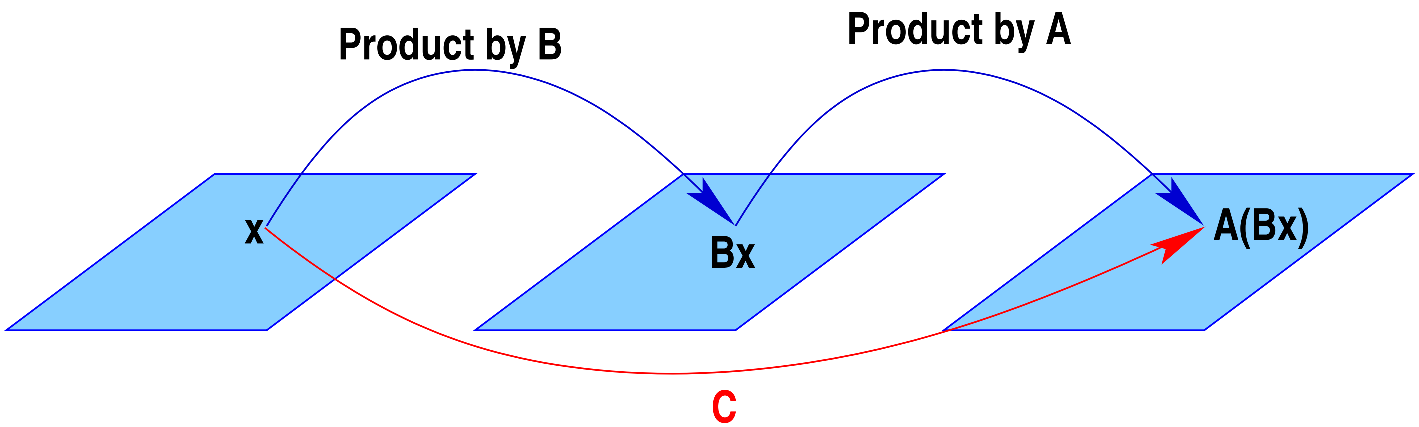
- Thus $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$ is produced from $\boldsymbol{x}$ by a composition of mappings– the linear transformations induced by $\boldsymbol{B}$ and $\boldsymbol{A}$.

- Note: $\boldsymbol{x} \rightarrow \boldsymbol{y}\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$ is a linear mapping (prove this).

Goal: to represent this composite mapping as a multiplication by a single matrix, call it C for now, so that

$$A(Bx) = Cx$$

.



- Assume A is $m \times n$, B is $n \times p$, and x is in $\mathbb{R}^p$
- Denote the columns of B by $b_1, \cdots, b_p$ and the entries in x by $x_1, \cdots, x_p$. Then:

$$Bx = x_1 b_1 + \cdots + x_p b_p$$

➤ By the linearity of multiplication by $\boldsymbol{A}$:

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{B}\boldsymbol{x}) &= \boldsymbol{A}(\boldsymbol{x}_1\boldsymbol{b}_1) + \cdots + \boldsymbol{A}(\boldsymbol{x}_p\boldsymbol{b}_p) \\ &= \boldsymbol{x}_1\boldsymbol{A}\boldsymbol{b}_1 + \cdots + \boldsymbol{x}_p\boldsymbol{A}\boldsymbol{b}_p \end{aligned}$$

➤ The vector $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$ is a linear combination of $\boldsymbol{A}\boldsymbol{b}_1, \cdots, \boldsymbol{A}\boldsymbol{b}_p$, using the entries in $\boldsymbol{x}$ as weights.

➤ In matrix notation, this linear combination is written as

$$\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x}) = [\boldsymbol{A}\boldsymbol{b}_1, \boldsymbol{A}\boldsymbol{b}_2, \cdots \boldsymbol{A}\boldsymbol{b}_p].\boldsymbol{x}$$

➤ Thus, multiplication by $[\boldsymbol{A}\boldsymbol{b}_1, \boldsymbol{A}\boldsymbol{b}_2, \cdots, \boldsymbol{A}\boldsymbol{b}_p]$ transforms $\boldsymbol{x}$ into $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{x})$.

➤ Therefore the desired matrix $\boldsymbol{C}$ is the matrix

$$\boldsymbol{C} = [\boldsymbol{A}\boldsymbol{b}_1, \boldsymbol{A}\boldsymbol{b}_2, \cdots, \boldsymbol{A}\boldsymbol{b}_p]$$

➤ Denoted by $\boldsymbol{A}\boldsymbol{B}$

Definition: If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $b_1, \cdots, b_p$, then the product AB is the matrix whose p columns are $Ab_1, \cdots, Ab_p$. That is:

$$AB = A[b_1, b_2, \cdots, b_p] = [Ab_1, Ab_2, \cdots, Ab_p]$$

➤ Important to remember that :

Multiplication of matrices corresponds to composition of linear transformations.

✍ Operation count: How many operations are required to perform product AB?

✍ Compute $\boldsymbol{AB}$ when

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

✍ Compute $\boldsymbol{AB}$ when

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 2 & 0 \\ 1 & -2 & 1 & 0 \\ 3 & -2 & 0 & 0 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -2 & 2 \\ 2 & 1 & -2 \\ -1 & 3 & 2 \end{bmatrix}$$

✍ Can you compute $\boldsymbol{AB}$ when

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ -1 & 4 \end{bmatrix}?$$

Row-wise matrix product

➤ Recall what we did with matrix-vector product to compute a single entry of the vector Ax

➤ Can we do the same thing here? i.e., How can we compute the entry c_{ij} of the product AB without computing entire columns?

✍ Do this to compute entry $(2, 2)$ in the first example above.

✍ Operation counts: Is more or less expensive to perform the matrix-vector product row-wise instead of column-wise?

Properties of matrix multiplication

Theorem Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

- $A(BC) = (AB)C$ (associative law of multiplication)
- $A(B + C) = AB + AC$ (left distributive law)
- $(B + C)A = BA + CA$ (right distributive law)
- $\alpha(AB) = (\alpha A)B = A(\alpha B)$ for any scalar α
- $I_m A = AI_n = A$ (product with identity)

✍ If $AB = AC$ then $B = C$ ('simplification') : True-False?

✍ If $AB = 0$ then either $A = 0$ or $B = 0$: True or False?

✍ $AB = BA$: True or false??

Square matrices. Matrix powers

- Important particular case when $n = m$ - so matrix is $n \times n$
- In this case if x is in $\mathbb{R}^n$ then $y = Ax$ is also in $\mathbb{R}^n$
- AA is also a square $n \times n$ matrix and will be denoted by A^2
- More generally, the matrix A^k is the matrix which is the product of k copies of A:

$$A^1 = A; \quad A^2 = AA; \quad \cdots \quad A^k = \underbrace{A \cdots A}_{k \text{ times}}$$

- For consistency define A^0 to be the identity: $A^0 = I_n$,

✍ $A^l \times A^k = A^{l+k}$ – Also true when k or l is zero.

Transpose of a matrix

Given an $m \times n$ matrix A, the transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

Theorem : Let A and B denote matrices whose sizes are appropriate for the following sums and products.

- $(A^T)^T = A$
- $(A + B)^T = A^T + B^T$
- $(\alpha A)^T = \alpha A^T$ for any scalar α
- $(AB)^T = B^T A^T$

More on matrix produts

➤ Recall: Product of the matrix $\boldsymbol{A}$ by the vector $\boldsymbol{x}$:

$$\overset{\boldsymbol{y}}{\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_j \\ \vdots \\ \beta_n \end{bmatrix}} = \overset{\boldsymbol{A}}{\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots a_{1n} \\ \vdots & & \vdots & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots a_{in} \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots a_{mn} \end{bmatrix}} \overset{\boldsymbol{x}}{\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_j \\ \vdots \\ \alpha_n \end{bmatrix}}$$

$$= \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$$

- $\boldsymbol{x}, \boldsymbol{y}$ are vectors; $\boldsymbol{y}$ is the result of $\boldsymbol{A} \times \boldsymbol{x}$.
- $a_1, a_2, ..., a_n$ are the columns of $\boldsymbol{A}$

- $\alpha_1, \alpha_2, ..., \alpha_n$ are the components of x [scalars]
- $\alpha_1 a_1$ is the first column of A multiplied by the scalar α_1 which is the first component of x.
- $\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n$ is a linear combination of $a_1, a_2, \cdots, a_n$ with weights $\alpha_1, \alpha_2, ..., \alpha_n$.

➤ This is the 'column-wise' form of the 'matvec'

Example: $$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \quad x = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix} \quad y = ?$$

➤ Result:

$$y = -2 \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \times \begin{bmatrix} 2 \\ -1 \end{bmatrix} - 3 \times \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -10 \end{bmatrix}$$

- Can get i-th component of the result y without the others:

$$\beta_i = \alpha_1 a_{i1} + \alpha_2 a_{i2} + \cdots + \alpha_n a_{in}$$

Example: In the above example extract β_2

$$\beta_2 = (-2) \times 0 + (1) \times (-1) + (-3) \times (3) = -10$$

- Can compute $\beta_1, \beta_2, \cdots, \beta_m$ in this way.
- This is the 'row-wise' form of the 'matvec'

Matrix-Matrix product

➤ When A is $m \times n$, B is $n \times p$, the product AB of the matrices A and B is the $m \times p$ matrix defined as

$$AB = [Ab_1, Ab_2, \cdots, Ab_p]$$

➤ Each Ab_j is a matrix-vector product: the product of A by the j-th column of B. Matrix AB has dimension $m \times p$

➤ Can use what we know on matvecs to perform the product

1. Column form – In words: *"The j-th column of AB is a linear combination of the columns of A, with weights $b_{1j}, b_{2j}, \cdots, b_{nj}$ "* (entries of j-th col. of B)

Example: $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -2 & 1 \\ 1 & -2 \\ -3 & 2 \end{bmatrix}$ $AB = ?$

Result:

$$B = \left[\begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \right]$$
$$= \begin{bmatrix} 3 & -6 \\ -10 & 8 \end{bmatrix}$$

- First column has been computed before: it is equal to:
(-2)*(col. 1 of A) + (1)*(col. 2 of A) + (-3)*(col. 3 of A)

- Second column is equal to:
(1)*(col. 1 of A) + (-2)*(col. 2 of A) + (2)*(col. 3 of A)

2. If we call C the matrix $C = AB$ what is c_{ij}? From above:

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ik}b_{kj} + \cdots + a_{in}b_{nj}$$

- Fix j and run $i \longrightarrow$ column-wise form just seen

3. Fix i and run $j \longrightarrow$ row-wise form

Example: Get second row of AB in previous example.

$$c_{2j} = a_{21}b_{1j} + a_{22}b_{2j} + a_{23}b_{3j}, \quad j = 1, 2$$

- Can be read as : $\boxed{c_{2:} = a_{21}b_{1:} + a_{22}b_{2:} + a_{23}b_{3:}}$, or in words:

row2 of C $= a_{21}$ (row1 of B) $+ a_{22}$ (row2 of B) $+ a_{23}$ (row3 of B)
$=$ 0 (row1 of B) $+$ (-1) (row2 of B) $+$ (3) (row3 of B)
$= [-10 \quad 8]$