VECTORS [PARTS OF 1.3]

Vectors and the set $\mathbb{R}^n$

- A vector of dimension n is an ordered list of n numbers

Example:

$$v = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}; \quad w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; z = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 4 \end{bmatrix}.$$

- v is in $\mathbb{R}^3$, w is in $\mathbb{R}^2$ and z is in $\mathbb{R}^?$

- In $\mathbb{R}^3$ the $\mathbb{R}$ stands for the set of real numbers that appear as entries in the vector, and the exponents 3, indicate that each vector contains 3 entries.

- A vector can be viewed just as a matrix of dimension $m \times 1$

- $\mathbb{R}^n$ is the set of all vectors of dimension n. We will see later that this is a vector space, i.e., a set that has some special properties with respect to operations on vectors.

- Two vectors in $\mathbb{R}^n$ are equal when their corresponding entries are all equal.

- Given two vectors u and v in $\mathbb{R}^n$, their sum is the vector $u+v$ obtained by adding corresponding entries of u and v

- Given a vector u and a real number α, the scalar multiple of u by α is the vector αu obtained by multiplying each entry in u by α

- (!) Note: the two vectors must be both in $\mathbb{R}^n$, i.e., then both have n components.

- Let us look at this in detail

Sum of two vectors

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}; \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}; \quad \rightarrow \quad \boldsymbol{x} + \boldsymbol{y} = \begin{bmatrix} x_1 + y_1 \\ y_2 + x_2 \\ x_3 + y_3 \end{bmatrix}$$

with numbers:

$$\boldsymbol{x} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}; \quad \boldsymbol{y} = \begin{bmatrix} 0 \\ 3 \\ -3 \end{bmatrix}; \quad \rightarrow \quad \boldsymbol{x} + \boldsymbol{y} = \begin{bmatrix} -1 \\ 5 \\ ?? \end{bmatrix}$$

Multiplication by a scalar

- Given: a number $\boldsymbol{\alpha}$ (a 'scalar') and a vector $\boldsymbol{x}$:

$$\boldsymbol{\alpha} \in \mathbb{R}, \quad \boldsymbol{x} \in \mathbb{R}^3, \rightarrow \boldsymbol{\alpha x} = \begin{bmatrix} \boldsymbol{\alpha x_1} \\ \boldsymbol{\alpha x_2} \\ \boldsymbol{\alpha x_3} \end{bmatrix}$$

with numbers:

$$\boldsymbol{\alpha = 4}; \quad \boldsymbol{x} = \begin{bmatrix} \mathbf{-1} \\ \mathbf{2} \\ \mathbf{3} \end{bmatrix} \rightarrow \boldsymbol{\alpha x} = \begin{bmatrix} \mathbf{-4} \\ \mathbf{8} \\ \mathbf{12} \end{bmatrix}$$

In the text vectors are represented by bold characters and scalars by light characters. We will often use Greek letters for scalars and regular latin symbols for vectors

Properties of + and $\alpha *$

➤ The vector whose entries are all zero is called the zero vector and is denoted by 0.

- (a) $\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{y} + \boldsymbol{x}$ (Addition is commutative)
- (b) $\boldsymbol{x} + (\boldsymbol{y} + \boldsymbol{z}) = (\boldsymbol{x} + \boldsymbol{y}) + \boldsymbol{z}$ (Addition is associative)
- (c) $\mathbf{0} + \boldsymbol{x} = \boldsymbol{x} + \mathbf{0} = \boldsymbol{x}$, ($\mathbf{0}$ is the vector of all zeros)
- (d) $\boldsymbol{x} + (-\boldsymbol{x}) = -\boldsymbol{x} + \boldsymbol{x} = \mathbf{0}$ ($-\boldsymbol{x}$ is the vector $(-\mathbf{1})\boldsymbol{x}$)
- (e) $\boldsymbol{\alpha}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{\alpha x} + \boldsymbol{\alpha y}$
- (f) $(\boldsymbol{\alpha} + \boldsymbol{\beta})\boldsymbol{x} = \boldsymbol{\alpha x} + \boldsymbol{\beta x}$
- (g) $(\boldsymbol{\alpha\beta})\boldsymbol{x} = \boldsymbol{\alpha}(\boldsymbol{\beta x})$
- (h) $\mathbf{1}\boldsymbol{x} = \boldsymbol{x}$ for any $\boldsymbol{x}$

Linear combinations

- Very important concept ..

A linear combination of m vectors is a vector of the form:

$$x = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_m x_m$$

where $\alpha_1, \alpha_2, \cdots, \alpha_m$, are scalars and $x_1, x_2, \cdots, x_m$, are vectors in $\mathbb{R}^n$.

- The scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$ are called the weights of the linear combination

- They can be any real numbers, including zero

Linear combinations

Example: Linear combinations of vectors in $\mathbb{R}^3$:

$$\boldsymbol{u} = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}; \quad \boldsymbol{w} = 2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$$

And we have:

$$\boldsymbol{u} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}; \quad \boldsymbol{w} = \begin{bmatrix} ? \\ ? \\ ? \end{bmatrix}$$

Note: for $\boldsymbol{w}$ the second weight is -1 and the third is $+1$.

The linear span of a set of vectors

Definition: If $v_1, \cdots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, \cdots, v_p$ is denoted by $\mathbf{span}\{v_1, \cdots, v_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $v_1, \cdots, v_p$. That is, $\mathbf{span}\{v_1, \cdots, v_p\}$ is the collection of all vectors that can be written in the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_p v_p$ with $\alpha_1, \alpha_2, \cdots, \alpha_p$ scalars.

✍ What is $\mathbf{span}\{u\}$ in $\mathbb{R}^2$ where $u = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

✍ What is $\mathbf{span}\{v\}$ in $\mathbb{R}^2$ where $v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$?

✍ What is $\mathbf{span}\{u, v\}$ in $\mathbb{R}^2$ with u, v given above?

✍ Does the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ belong to this $\mathbf{span}\{u, v\}$?

✍ Same question for the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

✍ What is $\mathbf{span}\{u, v\}$ in $\mathbb{R}^3$ when:

$$u = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \; v = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \; ?$$

✍ Do the vectors:

$$a = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}; \; b = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

belong to $\mathbf{span}\{u, v\}$ found in the previous question.?

✍ Is $\mathbf{span}\{u, v\}$ the same as $\mathbf{span}\{v, u\}$?

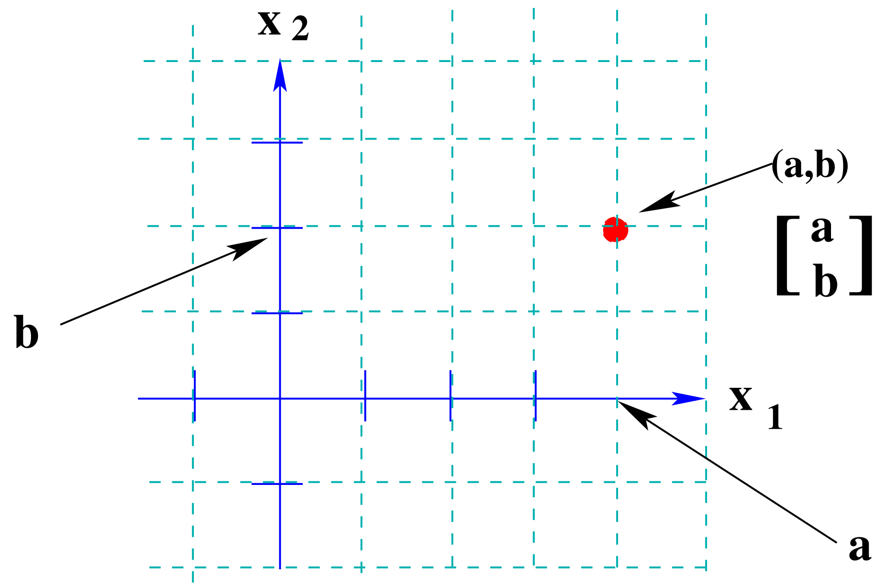
✍ Is $\mathbf{span}\{u, v\}$ the same as $\mathbf{span}\{2u, -3v\}$?

Geometric representation of $\mathbb{R}^2$ *and* $\mathbb{R}^3$

Consider a rectangular coordinate system in the plane. The illustration shows the vector

$$\boldsymbol{x} = \begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}$$

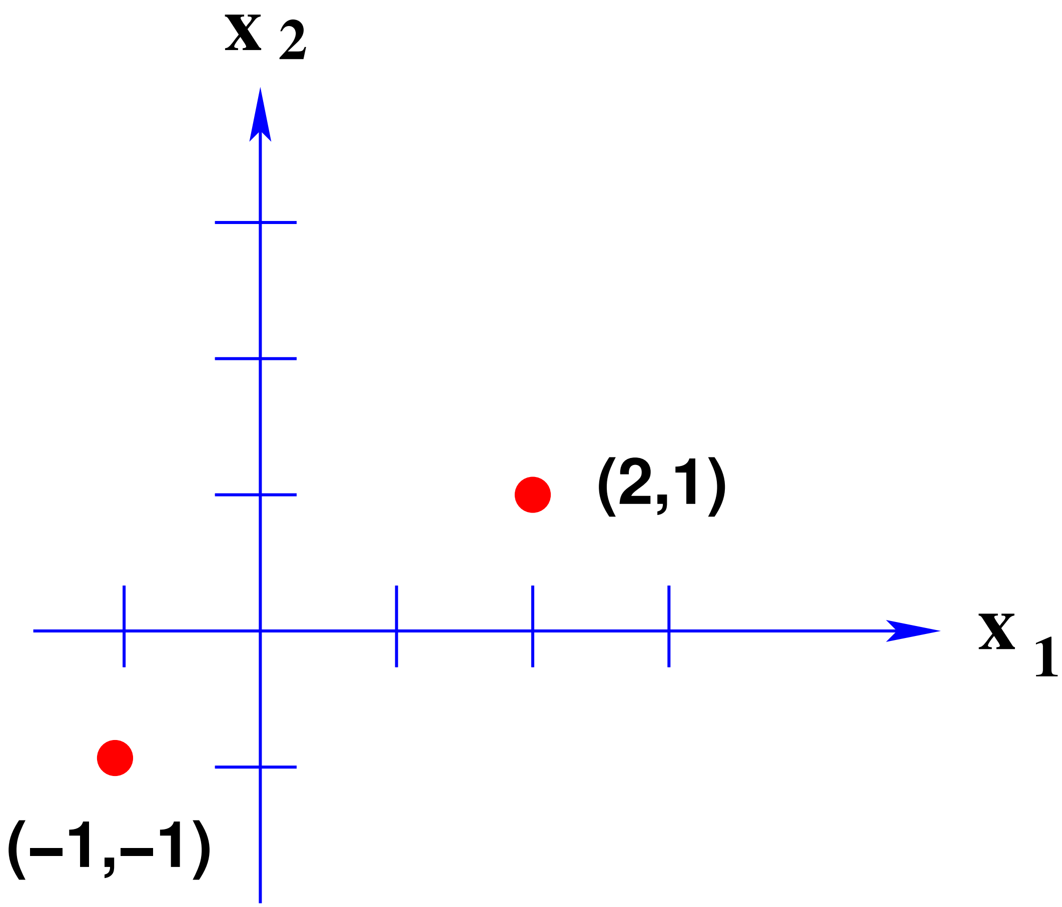
with $\boldsymbol{a = 4, b = 2}$.



➤ Each point in the plane is determined by an ordered pair of numbers, so we identify a geometric point $(\boldsymbol{a}, \boldsymbol{b})$ with the column vector $\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}$

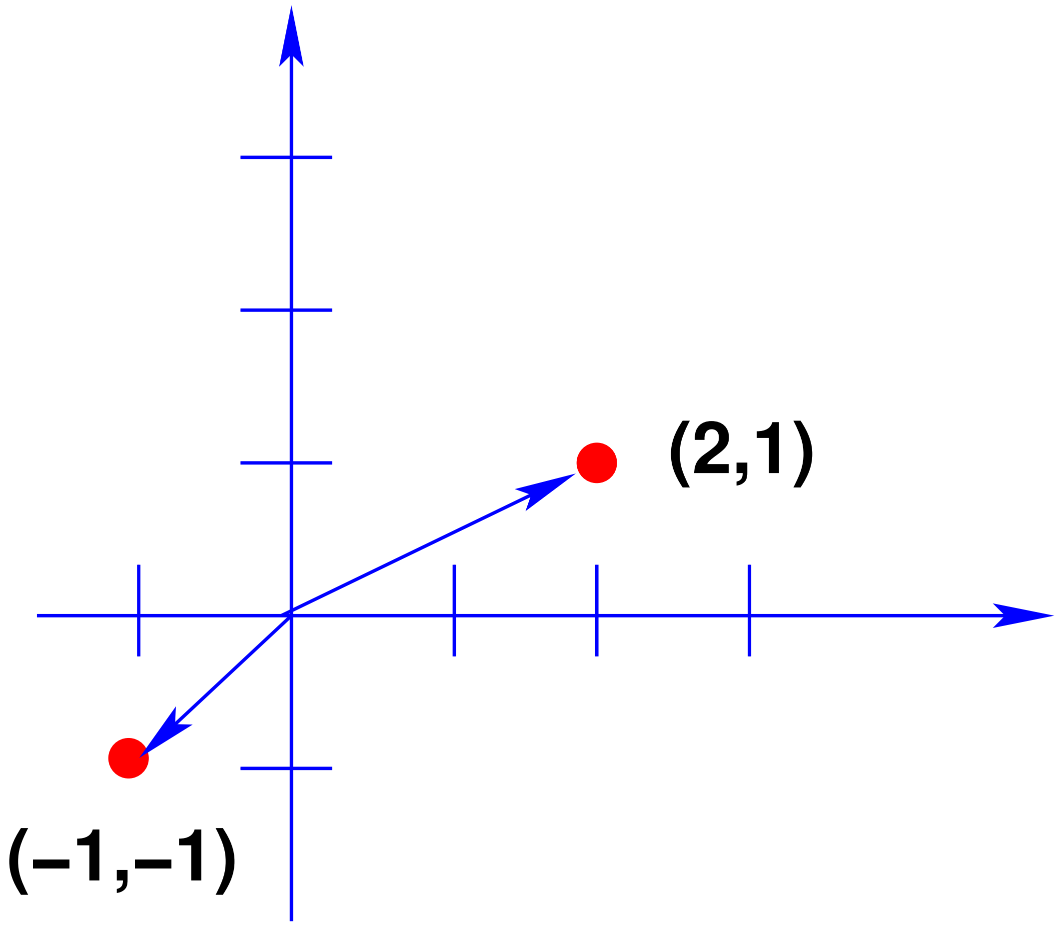
➤ We may regard $\mathbb{R}^2$ as the set of all points in the plane

$\mathbb{R}^2$

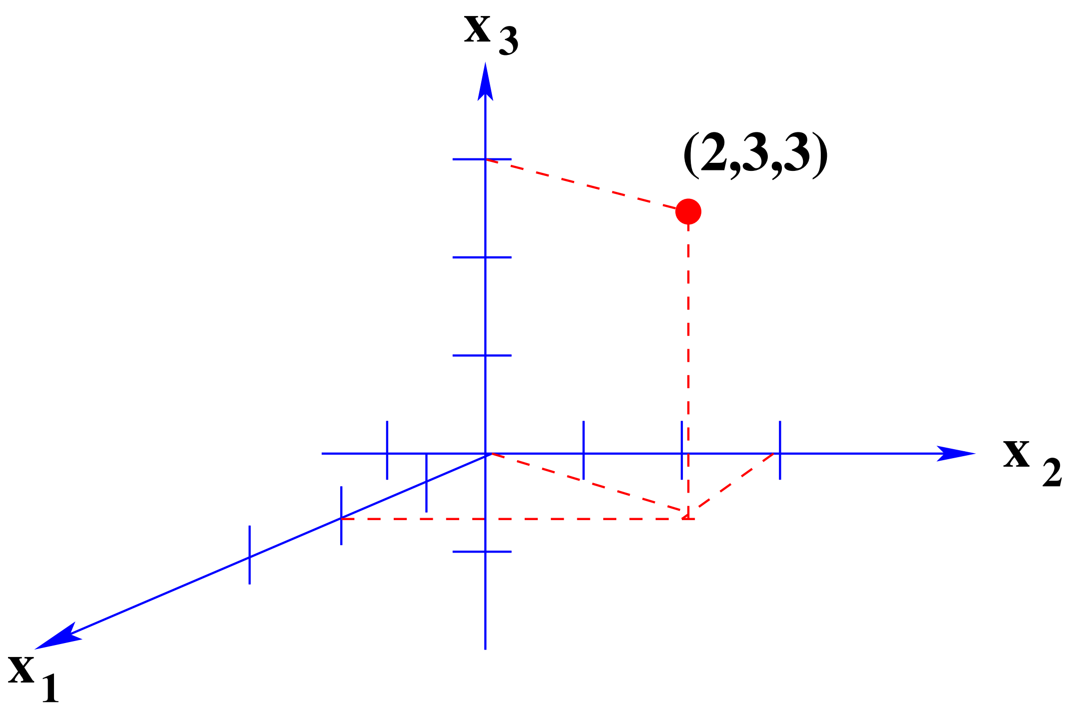


- x_1 in the horizontal direction, x_2 in vertical direction

➤ Often we draw an oriented line from origin to the point:



$\mathbb{R}^{\mathbf{3}}$

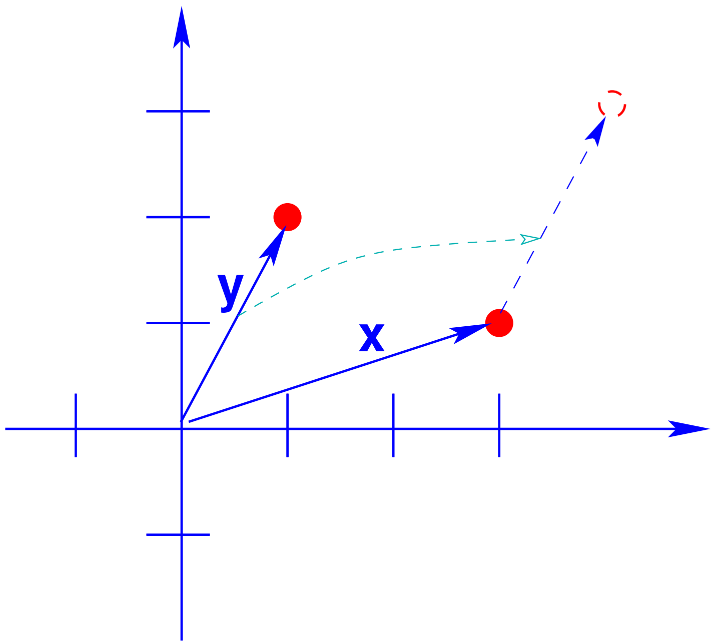


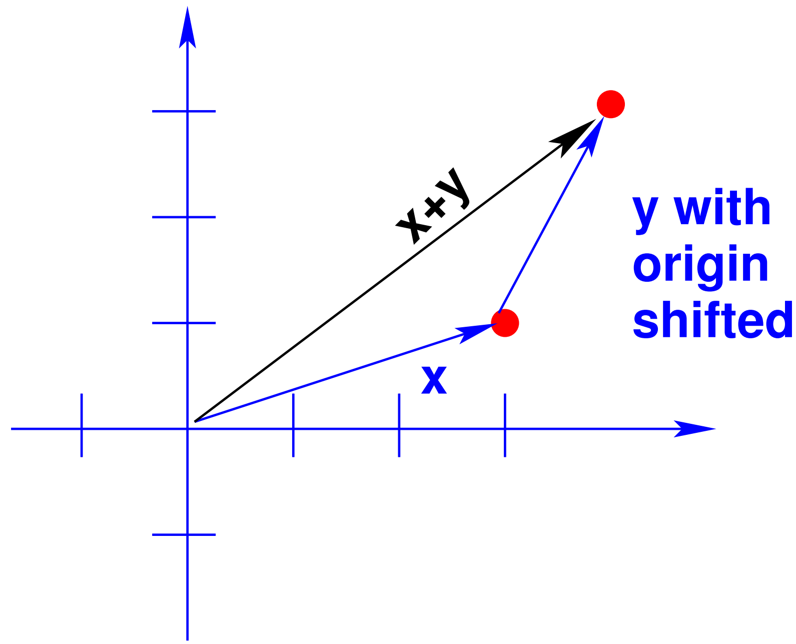
horizontal $= x_2$, vertical$=x_3$, back to front direction $= x_1$ (However some representations may differ). We will use this one.

Geometric interpretation of addition of 2 vectors

First viewpoint:

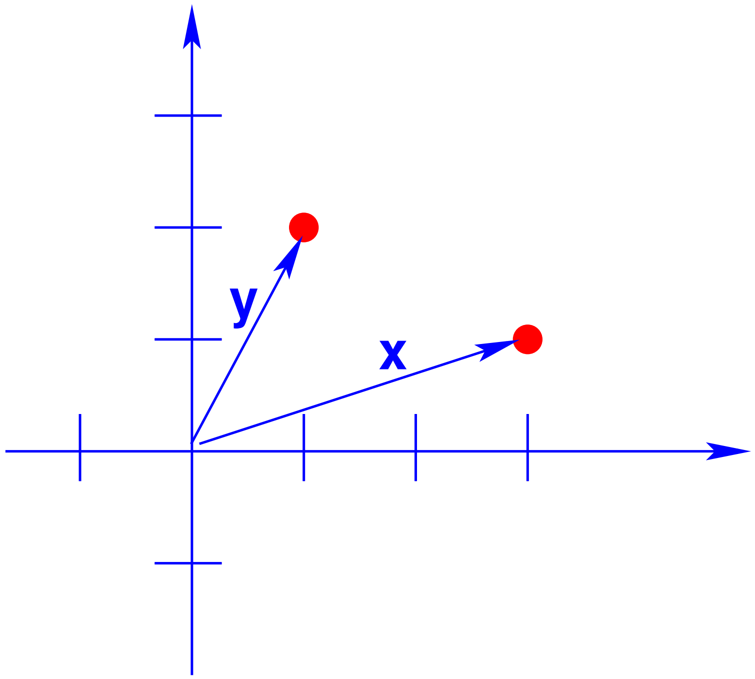
Think of moving ("rigidly") one of the vectors so its origin is at endpoint of the other vector. Then $\boldsymbol{x} + \boldsymbol{y}$ is the vector from origin to the end point of the shifted vector.

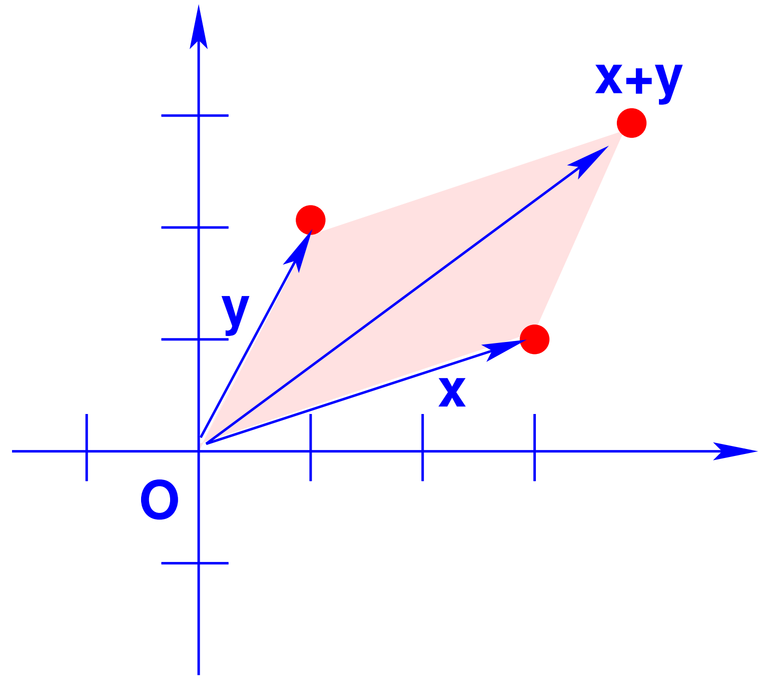




Second viewpoint:

$\boldsymbol{x} + \boldsymbol{y}$ correponds to the fourth vertex of the parallelogram whose other three vertices are: O, $\boldsymbol{x}$, and $\boldsymbol{y}$





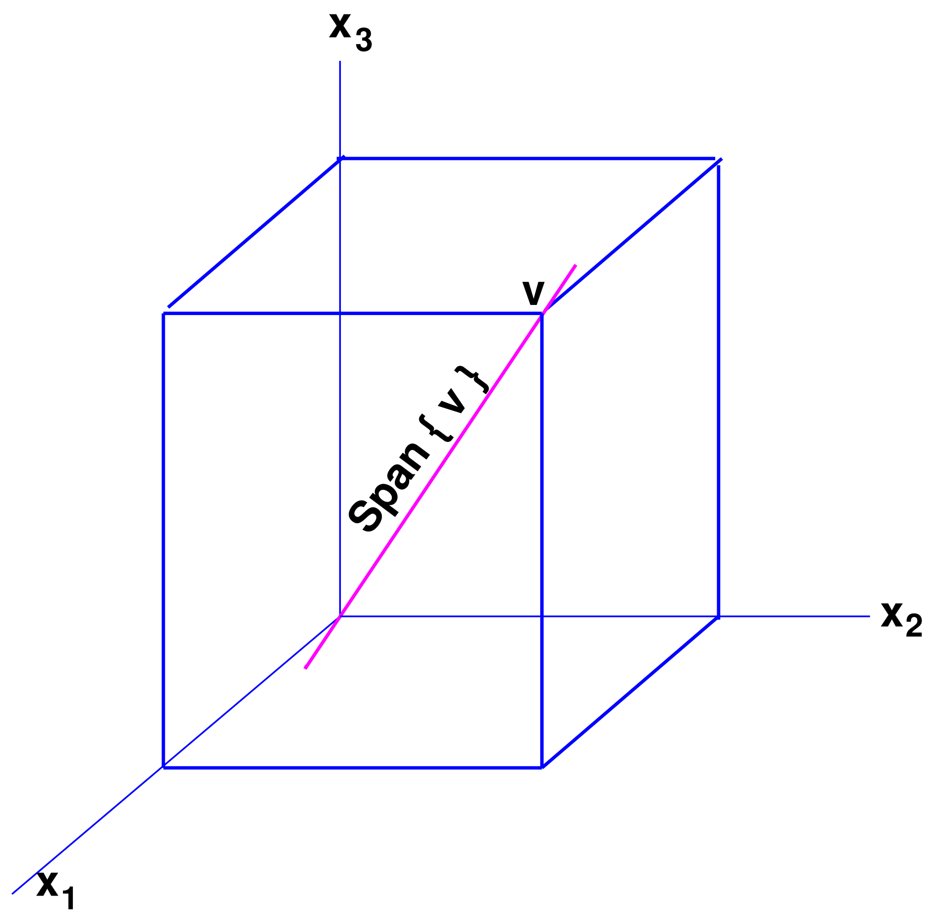
Using the first viewpoint, show geometrically how to add the 3 vectors $\begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$, $\begin{bmatrix} \mathbf{2} \\ \mathbf{0} \end{bmatrix}$, and $\begin{bmatrix} \mathbf{-1} \\ \mathbf{-2} \end{bmatrix}$

Geometric interpretation of span$\{v\}$

➤ Let $\boldsymbol{v}$ be a nonzero vector in $\mathbb{R}^3$

➤ Then $\mathbf{span}\{\boldsymbol{v}\}$ is the set of all scalar multiples of $\boldsymbol{v}$

➤ This is also the set of points on the line in $\mathbb{R}^3$ through $\boldsymbol{v}$ and $\mathbf{0}$.

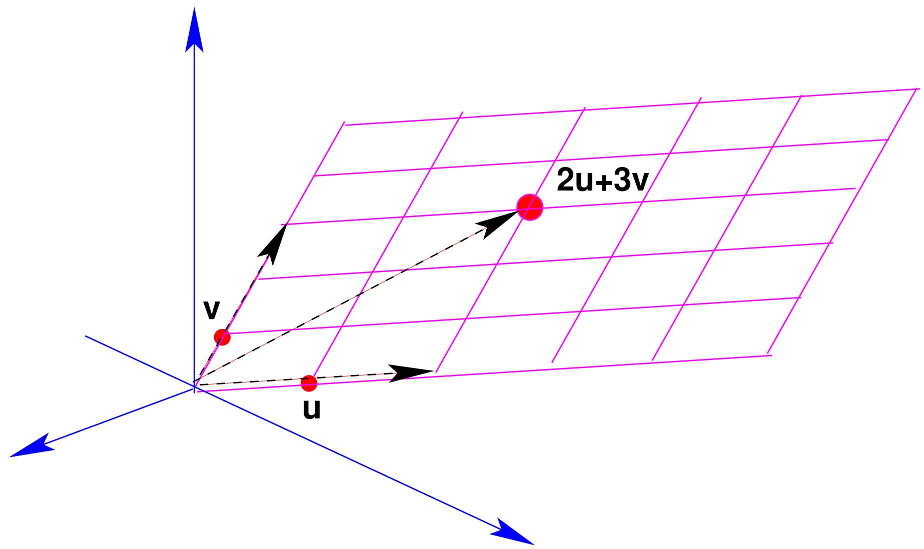


Geometric interpretation of $\text{span}\{u, v\}$

➤ Let $\boldsymbol{u}, \boldsymbol{v}$ be two nonzero vectors in $\mathbb{R}^3$ with $\boldsymbol{v}$ not a multiple of $\boldsymbol{u}$.

➤ Then $\textbf{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ is the plane in $\mathbb{R}^3$ that contains $\boldsymbol{u}, \boldsymbol{v}$, and $\mathbf{0}$.

➤ In particular, $\textbf{span}\{\boldsymbol{u}, \boldsymbol{v}\}$ contains the two lines $\textbf{span}\{\boldsymbol{u}\}$ and $\textbf{span}\{\boldsymbol{v}\}$



(See also Figure 1.1 from text).

LINEAR INDEPENDENCE [1.7]

Linear independence [Important]

Definition

➤ The set $\{v_1, ..., v_p\}$ is said to be linearly dependent if there exist weights $c_1, ..., c_p$, not all zero, such that

$$c_1 v_1 + c_2 v_2 + ... + c_p v_p = 0$$

➤ It is linearly independent otherwise

➤ The above equation is called linear dependence relation among the vectors $v_1, \cdots, v_p$

➤ Another way to express dependence: A set of vectors is linearly dependent if and only if there is one vector among them which is a linear combination of all the others.

✍ Prove this

Q: Why do we care about linear independence?

A: When expressing a vector $\boldsymbol{x}$ as a linear combination of a system $\{\boldsymbol{v}_1, \cdots, \boldsymbol{v}_p\}$ that is linearly dependent, then we can find a smaller system in which we can express $\boldsymbol{x}$

➤ A dependent system is 'redundant'

✍ Let $\boldsymbol{v}_1 = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$. Is $\{\boldsymbol{v}_1\}$ linearly independent? [here: $\boldsymbol{p} = \mathbf{1}$]

✍ A system consisting of a nonzero vector [at least one nonzero entry] is always linearly independent: True - False?

✍ Are the following systems linearly independent:

$$\left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{-10} \\ \mathbf{0} \end{bmatrix} \right\}, \quad \left\{ \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{-1} \\ \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{2} \\ \mathbf{-1} \end{bmatrix} \right\} ?$$

✍ A system $\{u, v\}$ is linearly dependent when ____________ ?

✍ Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$; $v_2 = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$; $v_3 = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$;

(a) Determine if $\{v_1, v_2, v_3\}$ is linearly independent

(b) If possible find a linear dependence relation among v_1, v_2, v_3.

Solution: We must determine if the system:

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

has a nontrivial solution (Trivial solution: $x_1 = x_2 = x_3 = 0$)

Augmented syst: Echelon 1st step Echelon 2nd step

$$\left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & 5 & 1 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & -3 & 5 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 4 & -2 & 0 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- This system is equivalent to original one.
- Variable x_3 is free.
- Select $x_3 = 3$ (to avoid fractions) and back-solve for x_2 ($x_2 = 5$), and x_1, ($x_1 = -14$)
- Conclusion: there is a nontrivial solution
- NOT independent

(b) Linear dependence relation: From above,

$$\boxed{-14v_1 + 5v_2 + v_3 = 0}$$

Note: Text uses the reduced echelon form instead of back-solving [Result is clearly the same. Both solutions are OK]

- With the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 14/3 & 0 \\ 0 & 1 & -5/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

- $x_1 = -(14/3)x_3; \quad x_2 = (5/3)x_3$
- select $x_3 = 3$ then $x_2 = 5, x_1 = 14$
- Recall: x_1, x_2 are basic variables, and x_3 is free