LINEAR EQATIONS [1.1] + (CONTINUED)

Gaussian Elimination

- Back to arbitrary linear systems.

Principle of the method: Since triangular systems are easy to solve, we will transform a linear system into one that is triangular. Main operation: combine rows so that zeros appear in the required locations to make the system triangular.

Recall Notation: Augmented form of a system

$$\begin{cases} 2x_1 + 4x_2 + 4x_3 = 2 \\ x_1 + 3x_2 + 1x_3 = 1 \\ x_1 + 5x_2 + 6x_3 = -6 \end{cases} \quad \text{Notation: } \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right]$$

- Main operation used: scaling and adding rows.

- Examples of such operations.

Example: Replace row 2 by: row 2 + row 1:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 3 & 7 & 5 & 3 \\ 1 & 5 & 6 & -6 \end{array}\right]$$

Example: Replace row 3 by: 2 times row 3 - row 1:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 3 & 7 & 5 & 3 \\ 1 & 5 & 6 & -6 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 3 & 7 & 5 & 3 \\ 0 & 6 & 8 & -14 \end{array}\right]$$

Example: Replace row 1 by: (0.5 * row 1)

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 3 & 7 & 5 & 3 \\ 0 & 6 & 8 & -14 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 3 & 7 & 5 & 3 \\ 0 & 6 & 8 & -14 \end{array}\right]$$

Gaussian Elimination (cont.)

- Go back to original system. Step 1 must eliminate x_1 from equations 2 and 3, i.e.,
- It must transform:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] \text{ into: } \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$$

$row_2 := row_2 - \frac{1}{2} \times row_1$: $\quad row_3 := row_3 - \frac{1}{2} \times row_1$:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right]$$

➤ Step 2 must now transform:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right] \text{ into: } \left[\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

$$row_3 := row_3 - 3 \times row_2 : \rightarrow \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right]$$

● System is now triangular

$$\begin{cases} 2x_1 + 4x_2 + 4x_3 = 2 \\ \quad\quad\ \ x_2 - x_3 = 0 \\ \quad\quad\quad\quad\ 7x_3 = -7 \end{cases} \rightarrow \text{Solve}$$

✍ Find the solution of the above triangular system and verify that it is a solution of the original system

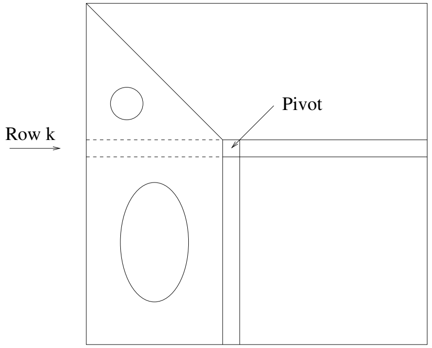
Gaussian Elimination: The algorithm

Recall: an algorithm is a sequence of operations (a 'recipe') to be performed by a computer.

➤ General step of Gaussian elimination :

➤ At step k subtract multiples of row k from rows $k+1, k+2, \cdots, n$ in order to zero-out entries below a_{kk} in column k.

➤ Repeat this step for $k = 1, 2, ..., n-1$



Step k in words:

for each row i where i runs from $i = k+1$ to $i = n$ do:
subtract piv * row k from row i (where $piv = a_{ik}/a_{kk}$).

ALGORITHM : 1. *Gaussian Elimination*

1. *For* $k = 1 : n-1$ *Do:*
2. *For* $i = k+1 : n$ *Do:*
3. $piv := a_{ik}/a_{kk}$
4. *For* $j := k+1 : n+1$ *Do :*
5. $a_{ij} := a_{ij} - piv * a_{kj}$
6. *End*
6. *End*
7. *End*

Matlab Script:

```
  function [x]  = gauss (A, b)
% function [x]  = gauss (A, b)
% solves A x  = b by Gaussian elimination
 n = size(A,1) ;
 A = [A,b];
 for k=1:n-1
    for i=k+1:n
      piv = A(i,k) / A(k,k) ;
      A(i,k+1:n+1)=A(i,k+1:n+1)-piv*A(k,k+1:n+1);
    end
 end
 x = backsolv(A,A(:,n+1));
```

➤ Input: matrix A and right-hand side b. Output: solution x.

➤ Invokes `backsolv.m` to solve final triangular system.

Gaussian Elimination: Pivoting

Consider again Gaussian Elimination for the linear system

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1 + 4x_2 + 6x_3 = -5 \end{cases}$$

Or:

2	2	4	2
1	1	1	1
1	4	6	−5

$row_2 := row_2 - \frac{1}{2} \times row_1$:

2	2	4	2
0	0	−1	0
1	4	6	−5

$row_3 := row_3 - \frac{1}{2} \times row_1$:

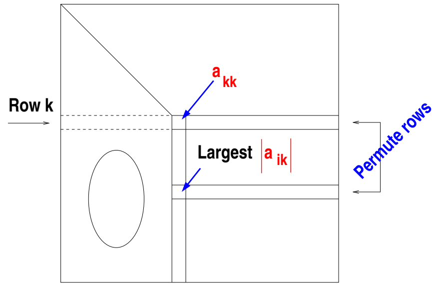
2	2	4	2
0	0	−1	0
0	3	4	−6

➤ Pivot a_{22} is zero. Solution : permute rows 2 and 3 ⟶

2	2	4	2
0	3	4	−6
0	0	−1	0

Gaussian Elimination: Partial Pivoting

General situation



➤ Partial Pivoting: *Always* Permute row k with row l such that

$$|a_{lk}| = \max_{i=k,\ldots,n} |a_{ik}|$$

➤ More 'stable' algorithm.

Gauss-Jordan Elimination

Principle of the method: We will now transform the system into one that is even easier to solve than a triangular system, namely a diagonal system. The method is very similar to Gaussian Elimination. It is just a bit more expensive.

Back to original system (P. 2-2). Step 1 must transform:

2	4	4	2
1	3	1	1
1	5	6	−6

into:

x	x	x	x
0	x	x	x
0	x	x	x

➤ Same step as 1st step of Gaussian Elimination.

$row_2 := row_2 - 0.5 \times row_1$: $row_3 := row_3 - 0.5 \times row_1$:

Step 1:

2	4	4	2
0	1	−1	0
1	5	6	−6

2	4	4	2
0	1	−1	0
0	3	4	−7

Must now transform:

2	4	4	2
0	1	−1	0
0	3	4	−7

into:

x	0	x	x
0	x	x	x
0	0	x	x

$row_1 := row_1 - 4 \times row_2$: $row_3 := row_3 - 3 \times row_2$:

Step 2:

2	0	8	2
0	1	−1	0
0	3	4	−7

2	0	8	2
0	1	−1	0
0	0	7	−7

There is now a third step:

To transform: $\left[\begin{array}{ccc|c} 2 & 0 & 8 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right]$ into: $\left[\begin{array}{ccc|c} x & 0 & 0 & x \\ 0 & x & 0 & x \\ 0 & 0 & x & x \end{array}\right]$

$row_1 := row_1 - \frac{8}{7} \times row_3$: $\quad row_2 := row_2 - \frac{-1}{7} \times row_3$:

Step 3: $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \end{array}\right]$ $\quad \left[\begin{array}{ccc|c} 2 & 0 & 0 & 10 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 7 & -7 \end{array}\right]$

Final System: $\begin{cases} 2x_1 = 10 \\ x_2 = -1 \\ 7x_3 = -7 \end{cases}$ Solution: $x_1 = 5$, $x_2 = -1$, $x_3 = -1$

Gauss-Jordan - variants

Common variant: Before an elimination step is started divide the row by diagonal entry a_{kk}

- At the end all diagonal entries are ones $\longrightarrow$ solution = rhs
- Redo the previous example with this variant.
- Is this more or less costly than the original method?

> NOTE: unless otherwise specified Gauss-Jordan will refer to this scaled version.

- Also: Pivoting can be implemented just like Gaussian elimination.

Important: Never swap a pivot row with a row above it! (destroys structure)


```
  function x = gaussj (A, b)
%---------------------------------------------
% function x = gaussj (A, b)
% solves A x  = b by Gauss-Jordan elimination
% this version scales rows.
%---------------------------------------------
 n = size(A,1) ;
 A = [A,b]  ;
 for k=1:n
    A(k,k:n+1) = A(k,k:n+1)/A(k,k);
    for i=1:n
        if (i ~= k)
            piv = A(i,k) ;
            A(i,k:n+1)=A(i,k:n+1)-piv*A(k,k:n+1);
        end
    end
 end
 x = A(:,n+1);
```


Linear systems – summary of complexity results

- The number of operations needed to solve a triangular linear system with n unknowns is

$$C_T(n) = n^2$$

- The number of operations required to solve a linear system with n unknowns by Gaussian elimination is

$$C_G(n) \approx \frac{2}{3}n^3$$

- The number of operations required to solve a linear system with n unknowns by Gauss-Jordan elimination is

$$C_{GJ}(n) \approx n^3$$

- Note: remember that Gauss-Jordan costs 50% more than Gauss.