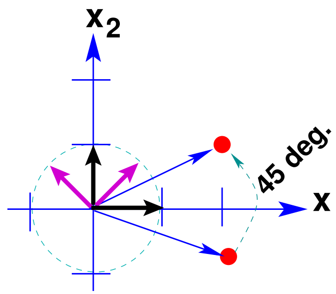
APPLICATION: ROTATION AND TRANSLATIONS [2.7]

Application: Rotations and translations in $\mathbb{R}^2$

➤ In the form of exercises. Try to answer all questions before class [see textbook for help]

✍ Consider the mapping that sends any point $\boldsymbol{x}$ in $\mathbb{R}^2$ into a point $\boldsymbol{y}$ in $\mathbb{R}^2$ that is rotated from $\boldsymbol{x}$ by an angle $\boldsymbol{\theta}$. Is the mapping linear?



✍ Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.]

[See Example 4 in Sect. 5.7 of *text* , See HW-2,..]

Solution: [see a previous HW]

➤ See how $\boldsymbol{e}_1$ are $\boldsymbol{e}_2$ are changed.

➤ $\boldsymbol{e}_1$ becomes $\boldsymbol{a}_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$

➤ $\boldsymbol{e}_2$ becomes $\boldsymbol{a}_2 = \begin{bmatrix} \cos(\pi/2+\theta) \\ \sin(\pi/2+\theta) \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$

➤ The columns of $\boldsymbol{A}$ are $\boldsymbol{a}_1, \boldsymbol{a}_2$; Therefore:

$$\boldsymbol{A} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotations and translations in $\mathbb{R}^2$

➤ Another very important operation: Translation or shift

➤ Recall: Not a linear mapping – but called affine mapping

➤ This will require a little artifice.

✍ How can you now represent a translation via a matrix-vector product? [Hint: add an artificial compoment of 1 at the end of vector $\boldsymbol{x}$]

➤ Called Homogeneous coordinates

➤ See Example 4 of Sect. 2.7 of *text* and then Example 6.

Solution: Call $f = [f_1; f_2]$ the translation vector

- Let $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$; Also write resulting vector $\hat{y}$ similarly as: $\hat{y} = \begin{bmatrix} y_1 \\ y_2 \\ 1 \end{bmatrix}$
- We want: $\hat{y}_1 = x_1 + f_1, \quad \hat{y}_2 = x_2 + f_2$
- Then the matrix is clearly: $A = \begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}$
- Indeed, we do have:

$$\begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 + f_1 \\ x_2 + f_2 \\ 1 \end{bmatrix} = \hat{y}$$

Rotations and translations in $\mathbb{R}^2$

- The most important mapping in real life is a combination of Rotation and Translation.
- Find a mapping that combines rotation followed by translation
- Hint: use the Homogeneous coordinates introduced above

Solution:

1. Rotation: Since this must leave the 1 at end of $\hat{x}$ unchanged the matrix is

$$R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2. Translation: The translation matrix is (see above)

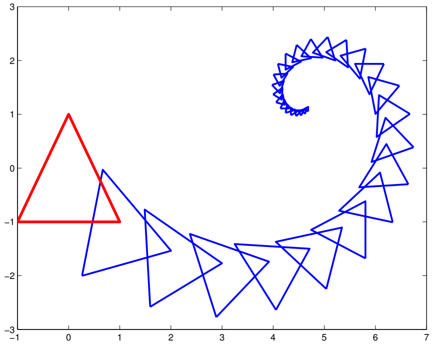
$$T = \begin{bmatrix} 1 & 0 & f_1 \\ 0 & 1 & f_2 \\ 0 & 0 & 1 \end{bmatrix}$$

3. Compound the two: This corresponds to product of matrices!

$$A = TR = \begin{bmatrix} \cos\theta & -\sin\theta & f_1 \\ \sin\theta & \cos\theta & f_2 \\ 0 & 0 & 1 \end{bmatrix}$$

- Does the order matter? Reason from the geometry and then from the derivation of your matrix
- One more operation: scaling by a weight α for example $\alpha = 0.3$. This corresponds to simply multiplying all coordinates by α
- See Composite transformations in text. See Example 6 in Sec. 2.7 in *text*. Implement the example in matlab [represent the triangle with vertices a=(-1, -1), b = (1, -1), c = (0,1). Ignore shading]

Practice. Continuing with Example 6 from *text* [previous exercise.] Generate the following figure using what you just learned.



Details: Scaling = 0.9; Rotation angle: $\theta = \pi/12$; Translation vector $(0.9, -0.9)$. Repeat: 30 times.

Challenge question: The triangles seem to vanish into a limit point. What is this point?