GENERAL VECTOR SPACES AND SUBSPACES [4.1]

General vector spaces

➤ So far we have seen special spaces of vectors of n dimensions – denoted by $\mathbb{R}^n$.

➤ It is possible to define more general vector spaces

A vector space V over $\mathbb{R}$ is a nonempty set with two operations:

- Addition denoted by $'+'$. For two vectors x and y, $x + y$ is a member of V
- Multiplication by a scalar For $\alpha \in \mathbb{R}$ and $x \in V$, αx is a member of V.

➤ In addition for V to be a vector space the following 8 axioms must be satisfied [note: order is different in text]

1. Addition is commutative $u + v = v + u$
2. Addition is associative $u + (v + w) = (u + v) + w$
3. $\exists$ zero vector denoted by 0 such that $\forall u$, $0 + u = u$
4. Any u has an opposite $-u$ such that $u + (-u) = 0$
5. $1u = u$ for any u
6. $(\alpha\beta)u = \alpha(\beta u)$
7. $(\alpha + \beta)u = \alpha u + \beta u$
8. $\alpha(u + v) = \alpha u + \alpha v$

✍ Show that the zero vector in Axiom 3 is unique, and the vector $-u$, ('negative of u'), in Axiom 4 is unique for each u in V.

➤ For each u in V and scalar α we have

$$0u = 0 \qquad \alpha 0 = 0 \; ; \qquad -u = (-1)u \; .$$

Example: Let V be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule, and for each v in V, define cv to be the arrow whose length is c times the length of v, pointing in the same direction as v if $c > 0$ and otherwise pointing in the opposite direction.

Note: The definition of V is geometric, using concepts of length and direction. No xyz-coordinate system is involved. An arrow of zero length is a single point and represents the zero vector. The negative of v is $(-1)v$.

➤ All axioms are verified

More examples

- Set of vectors in $\mathbb{R}^4$ with second component equal to zero.
- Set of all poynomials of degree ≤ 3
- Set of all $m \times n$ matrices
- Set of all $n \times n$ upper triangular matrices

Subspaces

- A subset H of vectors of V is a subspace if it is a vector space by itself. Formal definition:

- A subset H of vectors of V is a subspace if

1. H is closed for the addition, which means:

$$x + y \in H \quad \text{for any} \quad x \in H, y \in H$$

2. H is closed for the scalar multiplication, which means:

$$\alpha x \in H \quad \text{for any} \quad \alpha \in \mathbb{R}, x \in H$$

- Note: If H is a subspace then (1) 0 belongs to H and (2) For any $x \in H$, the vector $-x$ belongs to H

- Every vector space is a subspace (of itself and possibly of other larger spaces).
- The set consisting of only the zero vector of V is a subspace of V, called the zero subspace. Notation: $\{0\}$.

Example: Polynomials of the form

$$p(t) = \alpha_2 t^2 + \alpha_3 t^3$$

form a subspace of the space of polynomials of degree ≤ 3

Other examples: Examples 3 and 5 (sec. 4.1) from *text*

Example: Triangular matrices

Example 8 (sec. 4.1) in *text* is important

Show that the set H of all vectors in $\mathbb{R}^3$ of the form $\{a + b, a - b, b\}$ is a subspace of $\mathbb{R}^3$. [Hint: see example 11 from Sec. 4.1 of *text*]

- Recall: the term linear combination refers to a sum of scalar multiples of vectors, and $\text{span}\{v_1, ..., v_p\}$ denotes the set of all vectors that can be written as linear combinations of $v_1, \cdots, v_p$.

A subspace spanned by a set

Theorem: If $v_1, ..., v_p$ are in a vector space V, then

$$\text{span}\{v_1, ..., v_p\}$$

is a subspace of V.

➤ $\text{span}\{v_1, ..., v_p\}$ is the subspace spanned (or generated) by $\{v_1, ..., v_p\}$.

➤ Given any subspace H of V, a spanning (or generating) set for H is a set $\{v_1, ..., v_p\}$ in H such that $H = \text{span}\{v_1, ...v_p\}$.

✍ Prove above theorem for $p = 2$, i.e., given v_1 and v_2 in a vector space V, then $H = \text{span}\{v_1, v_2\}$ is a subspace of V. [Hint: show that H is closed for '+' and for scalar multiplication]

NULL SPACES AND COLUMN SPACES [4.2]

Null space of a matrix

Definition: The null space of an $m \times n$ matrix A, written as Nul(A), is the set of all solutions of the homogeneous equation $Ax = 0$. In set notation,

$$\text{Nul}(A) = \{x : x \in \mathbb{R}^n \quad \text{and} \quad Ax = 0\}.$$

Theorem: The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$

➤ Equivalently, the set of all solutions to a system $Ax = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$

Proof: Nul(A) is by definition a subset of $\mathbb{R}^n$. Must show: Nul(A) closed under $+$ and multipl. by scalars.

➤ Take u and v any two vectors in Nul(A). Then $Au = 0$ and $Av = 0$.

➤ Need to show that $u + v$ is in Nul(A), i.e., that $A(u + v) = 0$. Using a property of matrix multiplication, compute

$$A(u + v) = Au + Av = 0 + 0 = 0$$

➤ Thus $u + v \in$ Nul(A), and Nul(A) is closed under vector addition.

➤ Finally, if α is any scalar, then $A(\alpha u) = \alpha(Au) = \alpha(0) = 0$ which shows that αu is in Nul(A).

➤ Thus Nul(A) is a subspace of $\mathbb{R}^n$. ■

✍ See Example 1 in Sect. 4.2 of *text* [determining if a given vector belongs to Nul(A)

✍ See Example 2 in Sect. 4.2 of *text* [determining a subspace by casting as a null space]

➤ Next we will see how to determine Nul(A). See Example 3 of Sec. 4.2 of *text*. Details next.

- There is no obvious relation between vectors in Nul(A) and the entries in A.
- We say that Nul(A) is defined implicitly, because it is defined by a condition that must be checked.
- No explicit list or description of the elements in Nul(A), so..
- ... we need to solve the equation $Ax = 0$ to produce an explicit description of Nul(A).

Example: Find the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

- We will find a spanning set for Nul(A).

Solution: first step is to find the general solution of $Ax = 0$ in terms of free variables. We know how to do this.

- Get reduced echelon form of augmented matrix $[A\ 0]$:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{r} x_1 - 2x_2 \quad - x_4 + 3x_5 = 0 \\ x_3 + 2x_4 - 2x_5 = 0 \\ 0 = 0 \end{array}$$

- x_2, x_4, x_5 are free variables, x_1, x_3 basic variables.
- For any selection of the free variables, can find a vector in Nul(A) by computing x_1, x_3 in terms of these variables:

$$x_1 = 2x_2 + x_4 - 3x_5$$
$$x_3 = -2x_4 + 2x_5$$

- OK - but how can we write these using spanning vectors (i.e. as linear combinations of specific vectors?)
- Solution - write x as:

$$\begin{vmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{vmatrix} = \begin{vmatrix} 2x_2 & +x_4 & -3x_5 \\ x_2 & & \\ & -2x_4 & +2x_5 \\ & x_4 & \\ & & x_5 \end{vmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w}$$

- General solution is of the form $x_2 u + x_4 v + x_5 w$.
- Every linear combination of u, v, and w is an element of Nul(A). Thus $\{u, v, w\}$ is a spanning set for Nul(A), i.e.,

$$\text{Nul}(A) = \text{span}\{u, v, w\}$$

✍ Obtain the vector x of Nul(A)corresponding to the choice: $x_2 = 1, x_4 = -2, x_5 = -1$. Verify that indeed it is in the null space, i.e., that $Ax = 0$

✍ For same example, find a vector in Nul(A)whose last two components are zero and whose first component is 1. How many such vectors are there (zero, one, or inifintely many?)

Notes:

- 1. The spanning set produced by the method in the example is guaranteed to be linearly independent

✍ Show this (proof by contradiction)

- 2. When Nul(A)contains nonzero vectors, the number of vectors in the spanning set for Nul(A) equals the number of free variables in the equation $Ax = 0$.

Column Space of a matrix

Definition: The column space of an $m \times n$ matrix A, written as Col(A) (or $C(A)$), is the set of all linear combinations of the columns of A. If $A = [a_1 \cdots a_n]$, then

$$\text{Col}(A) = \text{span}\{a_1, ..., a_n\}$$

Theorem: The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

➤ A vector in Col(A) can be written as Ax for some x [Recall that Ax stands for a linear combination of the columns of A].

That is: $$\text{Col}(A) = \{b : b = Ax \quad \text{for some } x \text{ in } \quad \mathbb{R}^n\}$$

➤ The notation Ax for vectors in Col(A) also shows that Col(A) is the range of the linear transformation $x \to Ax$.

➤ The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $Ax = b$ has a solution for each b in $\mathbb{R}^m$

✍ Let

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad u = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}, \quad v = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

a. Determine if u is in Nul(A). Could u be in Col(A)?

b. Determine if v is in Col(A). Could v be in Nul(A) ?

➤ General remarks and hints:

1. Col(A) is a subspace of $\mathbb{R}^m$ [$m = 3$ in above example]

2. Nul(A) is a subspace of $\mathbb{R}^n$ [$n = 4$ in above example]

3. To verify that a given vector x belongs to Nul(A) all you need to do is check if $Ax = 0$

4. To verify if $b \in$ Col(A) all you need to do is check if the linear system $Ax = b$ has a solution.