CSci 2033, S'18	**Homework # 4**	**Due Date: 04/25/2018**

1. Find the volume of the paralleliped with one vertex at the origin and adjacent vertices $(1, 3, 0), (-2, 0, 2)$ and $(-1, 3, -1)$.

2. (a) Determine the range (column space) and null space of the following matrix:

$$\begin{bmatrix} 1 & 4 & 3 & -2 \\ -1 & -1 & 3 & -1 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

   (b) Does the vector $v = [1; 0; 1; 2]$ (matlab notation) belong to the null space of A?

   (c) Does the vector $v = [1; 2; -1; 1]$ (matlab notation) belong to the range of A?

   (d) Does the vector $v = [1; 1; 1; 1]$ (matlab notation) belong to the range of A? [Hint: See practice exercise 1 preceding set 2.8 of exercises in text. P. 152 of 5th edition. You can use the standard or reduced echelon forms which you can compute 'by hand'].

3. Is it true that the column space of AB is contained in the column space of A (justify your answer)? Give an example where the column spaces of AB and A are different.

4. Let $H = \textbf{span}\{v_1, v_2, v_3\}$ with

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}; \quad v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad v_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix};$$

   Show that v_3 is a linear combination of the first 2 vectors and then find a basis of H.

5. Find bases for Col(A) and for Nul(A) when A is the following matrix:

$$\begin{bmatrix} 1 & -2 & 3 & -5 & 7 \\ 0 & 0 & 2 & -3 & 4 \\ -1 & 2 & -1 & 2 & -3 \\ 1 & -2 & 5 & -8 & 11 \end{bmatrix}$$

   [Hint: You may use Matlab to extract the RREF form of the matrix] What is the rank of A?

6. Find the rank of A and also the rank of A^T [note that for the second matrix the rank depends on the parameter q] –

$$A = \begin{pmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{pmatrix}$$

7. Find the least-squares solution to the system $\min \|b - Ax\|$ where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \\ 0 & -1 \\ -1 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

Find the QR factorization of the matrix A. See the lecture notes and the text (theorem 15 in Section 6.5) to find out how to solve least-squares systems using the QR factorization. Find the solution to the least squares problem using this approach. [Note: as it turns out, because of rounding errors, this approach is numerically more reliable than the one based on normal equations.]

8. The logistic law of populations states that a given population (e.g. world population, or population of a given country, or state) evolves like $p(t) = ap_0/[bp_0 + (a - bp_0)e^{-a(t-t_0)}]$ where $p_0 = p(t_0)$ is the initial population at time t_0. Note that the inverse of p is a constant plus a scalar*exponential.

Based on this, we will seek to estimate the **inverse** of the US population by the formula:

$$q(t) = a_1 e^{-0.016*(t-t_n)} + a_2 e^{-0.020*(t-t_n)} + a_3 e^{-0.024*(t-t_n)}$$

Here, t represents time (in years), t_n represents the last year available (in our case $t_n = 2010$), a_1, a_2, a_3, are 3 scalars to be found, and $q(t)$ is an estimate of the inverse of $p(t)$.

Calculate a_1, a_2, a_3 by least-squares. The population data for every year the census is conducted (from year 1800) is posted in the class web-site. Give the corresponding estimate for $q(t)$ and then for $p(t) = 1/q(t)$ the population for the year March 30, 2018. (a 1/4 of one year into 2018, i.e., $y = 2018.25$) Compare with the census estimate located here: `https://www.census.gov/en.html`

What is your estimate of the US population for year 2050?

9. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{pmatrix}$$

   (a) Find the eigenvalues and associated eigenvectors of A ;

   (b) Is A a diagonalizable matrix? If so find a matrix M such that $M^{-1}AM$ is diagonal. Is M unique? Explain.

   (c) Find the eigenvalues and associated eigenvectors of A^2.

   (d) Show that $A + I$ is invertible and find the eigenvalues and associated eigenvectors of $(A + I)^{-1}$.

   (e) Let $B = A/3$. Determine $9B^{213}$ and $9B^{214}$ (computing the results with matlab in a brute force manner will be discarded as incorrect)

10. Let $A = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$. Compute A^{16} by successive squaring (i.e., set $B = A^2$, then apply the recurrence $B := B^2$ a few times to get A^{16}). Then compute A^{16} by using the eigenvalue decomposition of A. (hint: exploit the fact that $(1/2)^{16}$ is very small).

11. Note: this question is now treated as an extra-credit question [5 points of extra credit out of 100]. It will be necessary to do some reading of the text and notes on material that has not yet been covered.

    You will find in the matlab section of the course website a matrix called X and stored in a file called `X.mat` (load with the command `load('X')`). Using matlab compute the Singular Value Decomposition of the matrix X. What is the approximate rank r of the matrix?

    Find the subspace of dimension r that represents the column space associated with this rank.