CSci 2033, S'18	**Homework # 3**	**Due Date: 04/04/2018**

1. (a) Prove or disprove: If the columns of $B \in \mathbb{R}^{n\times p}$ are linearly *dependent* then so are the columns of AB (for any $A \in \mathbb{R}^{m\times n}$).

   (b) Prove or disprove: If the columns of $B \in \mathbb{R}^{n\times p}$ are linearly *independent* as well as those of A, then so are the columns of AB (for $A \in \mathbb{R}^{m\times n}$).

2. Use the two existence theorems in Page 6-8 and Page 8-15 of the notes to prove the following statement: If Gaussian Elimination *with partial pivoting* for solving $Ax = b$ breaks down then A is not invertible. [Hints: 1) Pivot positions for the rref and standard row echelon form (ref) are the same (explain); 2) GE fails only when ..... 3) Then the theorem in p. 6-8 will show that $Ax = b$ does not always have a solution; 4) Conclude by using the theorem in Page 8-15.]

3. What are the elimination matrices E_1 (step 1), E_2 (step 2) that transform the following matrix into upper triangular matrix U by Gaussian elimination? Find the inverses of these matrices and multiply them to find L such that $A = LU$.
$$A = \begin{bmatrix} -2 & 4 & 1 \\ 1 & -1 & -1 \\ 1 & -3 & -2 \end{bmatrix}$$

4. Find the inverse of the matrix A below by using the cofactor formula (P. 10-12 of notes):
$$A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

5. Write a sequence of 3 matlab commands to compute the inverse of a matrix using `rref`. The first of these lines is: `n = size(A,1)`. The second should invokes matlab's `rref` on a certain matrix. The third line will extract the inverse from the result of `rref`. Apply your 3-line code to find the inverse of the same matrix A as in question 3. [Show your final short matlab program: one line to define the matrix, e.g., `A=[ -2 4 1; 1 -1 -1; 1 -3 -2]`, followed by the 3 lines discussed above. Then show the result of this 4-line program]

6. The matrix on the right has a very simple inverse. Find A^{-1} for example with matlab. Extend this to a five by five matrix of the same type (upper triangular matrix with diagonals of 1s and -1s alternatingly). Next, prove the result for a general matrix of this type of dimension $n \times n$?
$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
   [Hint: consider AB where B is your (guessed) inverse. What can you say about the j-th column of AB for any $j = 1, 2, \cdots, n$?]

7. (a) Use determinants to decide if or when (for C) the following matrices are invertible:
$$A = \begin{bmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{bmatrix}; \quad B = \begin{bmatrix} 3 & 4 & -2 \\ 2 & 5 & 5 \\ 1 & 2 & 1 \end{bmatrix}; \quad C = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 5 & 3 \\ 3 & \alpha & 4 \end{bmatrix};$$

   (b) For C find how you can reach the same conclusion by using the LU factorization.

8. (a) What are the LU factors of the matrices

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}; \quad B = \begin{bmatrix} a & a & 0 \\ a & a+b & b \\ 0 & b & b+c \end{bmatrix}.$$

(b) Compute the determinants of A and B (use your result of (a)). Show that B is invertible when a, b, c are all nonzero. By inspecting the columns of B, can you say why is B not invertible when $a = 0$? Why is it not invertible when $b = 0$? Why is it not invertible when $c = 0$?

(c) **Extra credit: 5 points** The matrices in (a) are tridiagonal. What are the shapes of the factors L and U for an $n \times n$ tridiagonal matrix? What is the cost the LU factorization in this case?

9. (a) Find the area of the parallelogram with edges $u = (2, 1)$ and $v = (-1, 3)$.
(b) This parallelogram is now transformed into another parallelogram by the matrix

$$A = \begin{bmatrix} 2 & -1 \\ -1 & -1 \end{bmatrix}$$

Show a plot of the two parallelograms on the same figure [Hint: You need only show how the vertices are transformed. You may use matlab for this [or do the plot by hand]. What is the area of this new parallelogram?
(c) How can obtain this area from the determinant of A and the area of the original parallelogram?

10. (Matlab) Develop a script to perform the LU factorization (without pivoting) whose calling sequence is of the form

```
function [L U] = myLU (A)
```

You can start from the Gaussian elimination script `gauss` posted earlier. A 5×5 matrix will be posted in the class web site for testing your script.
Provide: a print-out of your script along with the L and U matrices you obtained from running it on the 5×5 matrix A provided.