Kalman Filter
(Ch. 15)

Kalman Filters

How does all of this relate to Kalman filters?

This is just “filtering” (in HMM/Bayes net), except with continuous variables

This heavily use the Gaussian distribution:

$$N(\mu, \sigma^2)(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-1}{2}\frac{(x-\mu)^2}{\sigma^2}} = \alpha e^{\frac{-1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

thank you alpha!

Kalman Filters

Why the preferential treatment for Gaussians?

A key benefit is that when you do our normal operations (add and multiply), if you start with a Gaussian as input, you get Gaussian out

In fact, if you input a linear Gaussian input, you get a Gaussian out: (linear=matrix mult)

More on this later, let's start simple

Kalman Filters

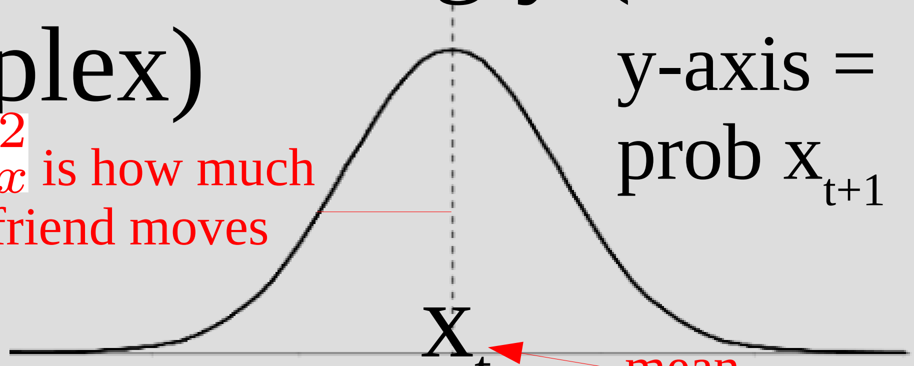
As an example, let's say you are playing Frisbee at night

1. Can't see exactly where friend is

2. Friend will move slightly to catch Frisbee

Kalman Filters

Unfortunately... the math is a bit ugly (as Gaussians are a bit complex)



Here we assume:

$$P(x_{t+1}|x_t) = \alpha e^{\frac{-1}{2}\frac{(x_{t+1}-x_t)^2}{\sigma_x^2}} = N(x_t, \sigma_x^2)(x_{t+1})$$

$$P(e_t|x_t) = \alpha e^{\frac{-1}{2}\frac{(e_t-x_t)^2}{\sigma_e^2}} = N(x_t, \sigma_e^2)(e_t)$$

variance is "can't see well"

How do we compute the filtering "forward" messages (in our efficient non-recursive way)?

Kalman Filters

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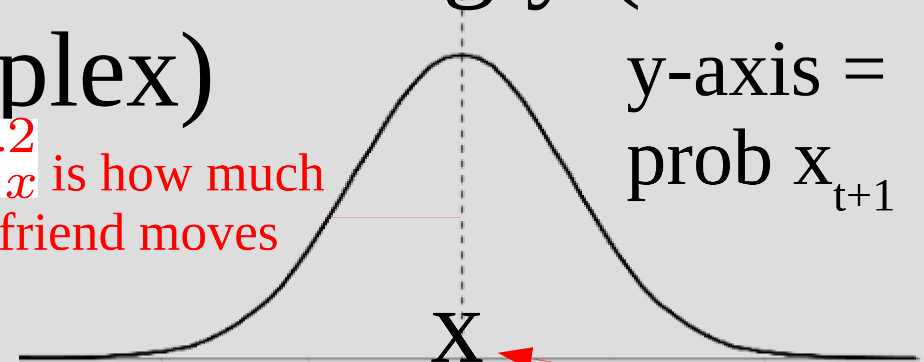

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variance is “can’t see well”

erm... let’s change variable names

How do we compute the filtering “forward” messages (in our efficient non-recursive way)?

Kalman Filters

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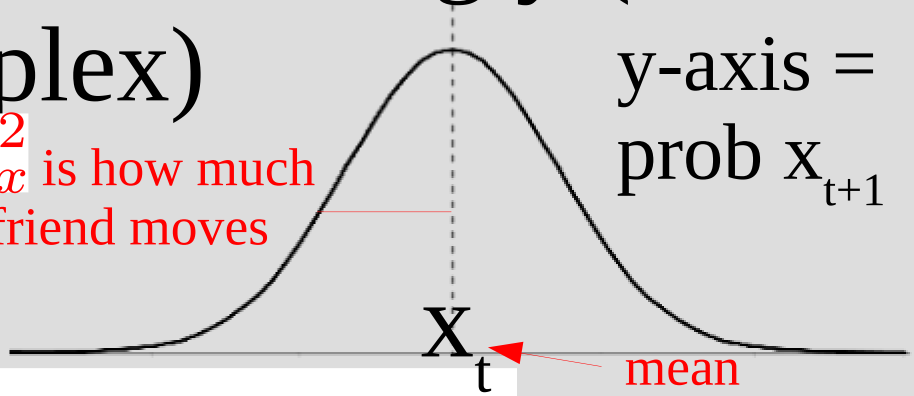

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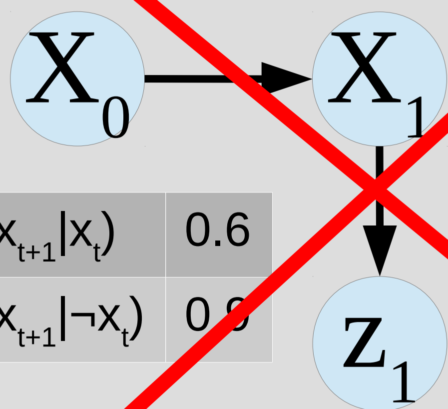
variance is “can’t see well”

How do we compute the filtering “forward” messages (in our efficient non-recursive way)?

Kalman Filters

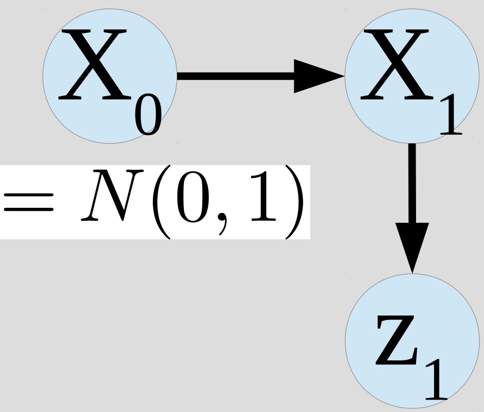


$P(x_{t+1}|x_t)$	0.6
$P(x_{t+1}|\neg x_t)$	0.9

$P(e_t|x_t)$	0.3
$P(e_t|\neg x_t)$	0.8

$$P(x_{t+1}|x_t) = N(x_t, \sigma_x^2)(x_{t+1})$$

$$P(x_0) = N(0,1)$$



$$P(z_t|x_t) = N(x_t, \sigma_z^2)(z_t)$$

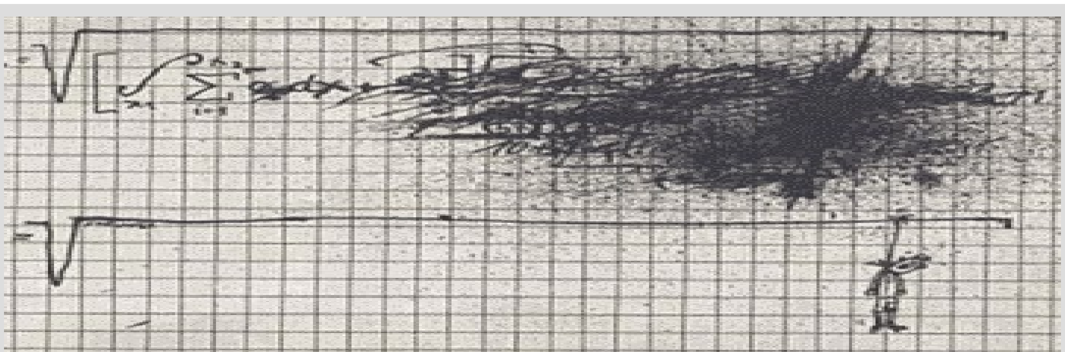
The same? Sorta... but we have to integrate

$$F_1 = P(z_1|x_1) \sum_{x_0} P(x_1|x_0)P(x_0)$$

$$F_1 = P(z_1|x_1) \int_{-\infty}^{\infty} P(x_1|x_0)P(x_0)dx_0$$

Kalman Filters

$$
\begin{aligned}
F_1 &= P(z_1|x_1)\int_{-\infty}^{\infty} P(x_1|x_0)P(x_0)dx_0 \\
&= N(x_1,\sigma_z^2)(z_1)\int_{-\infty}^{\infty} N(x_0,\sigma_x^2)(x_1)\cdot N(0,1)(x_0)dx_0 \\
&= N(x_1,\sigma_z^2)(z_1)\int_{-\infty}^{\infty} \alpha e^{\frac{-1}{2}\frac{(x_1-x_0)^2}{\sigma_x^2}}\cdot \alpha' e^{\frac{-1}{2}x_0^2}dx_0 \\
&= \hat{\alpha}N(x_1,\sigma_z^2)(z_1)\int_{-\infty}^{\infty} e^{\frac{-1}{2}\frac{(x_1-x_0)^2+\sigma_x^2x_0^2}{\sigma_x^2}}dx_0 \\
&= \hat{\alpha}N(x_1,\sigma_z^2)(z_1)\int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{(1+\sigma_x^2)}{\sigma_x^2}x_0^2+\frac{(-2x_1)}{\sigma_x^2}x_0+\frac{x_1^2}{\sigma_x^2}\right)}dx_0
\end{aligned}
$$



Kalman Filters

But wait! There's hope!

We can use a little fact that:

$$ax^2 + bx + c = a(x - \frac{-b}{2a})^2 + \underbrace{(c - \frac{b^2}{4a})}_{\text{does not contain } x}$$

$$\begin{aligned} F_1 &= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) \int_{-\infty}^{\infty} e^{\frac{-1}{2}\left(\frac{(1+\sigma_x^2)}{\sigma_x^2} x_0^2 + \frac{(-2x_1)}{\sigma_x^2} x_0 + \frac{x_1^2}{\sigma_x^2}\right)} \\ &= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2 + (c - \frac{b^2}{4a})} dx_0 \\ &= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2} e^{\frac{-1}{2}(c - \frac{b^2}{4a})} dx_0 \\ &= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}(c - \frac{b^2}{4a})} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2} dx_0 \end{aligned}$$

Kalman Filters

But wait! There's hope!

We can use a little fact that:

$$ax^2 + bx + c = a(x - \frac{-b}{2a})^2 + \underbrace{(c - \frac{b^2}{4a})}_{\text{does not contain } x}$$

$$F_1 = \hat{\alpha} N(x_1, \sigma_z^2)(z_1) \int_{-\infty}^{\infty} e^{\frac{-1}{2}(\frac{(1+\sigma_x^2)}{\sigma_x^2} x_0^2 + \frac{(-2x_1)}{\sigma_x^2} x_0 + \frac{x_1^2}{\sigma_x^2})}$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2 + (c - \frac{b^2}{4a})} dx_0$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2} e^{\frac{-1}{2}(c - \frac{b^2}{4a})} dx_0$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}(c - \frac{b^2}{4a})} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2} dx_0$$

This is just: $N(\frac{-b}{2a}, \frac{1}{a})$

Kalman Filters

$$a = \frac{1+\sigma_x^2}{\sigma_x^2}, b = \frac{-2x_1}{\sigma_x^2}, c = \frac{x_1^2}{\sigma_x^2}$$

$$F_1 = \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}(c - \frac{b^2}{4a})} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(a(x_0 - \frac{-b}{2a})^2} dx_0$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}(c - \frac{b^2}{4a})} \int_{-\infty}^{\infty} N(\frac{-b}{2a}, \frac{1}{a})(x_0) dx_0$$

area under all of normal distribution adds up to 1

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}(c - \frac{b^2}{4a})} \cdot 1$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}\left(\frac{x_1^2}{\sigma_x^2} - \frac{(\frac{-2x_1}{\sigma_x^2})^2}{4\frac{1+\sigma_x^2}{\sigma_x^2}}\right)} \cdot 1$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}\left(\frac{x_1^2}{\sigma_x^2} - \frac{4x_1^2/\sigma_x^4}{4(1+\sigma_x^2)/\sigma_x^2}\right)} \cdot 1$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}\left(\frac{x_1^2(1+\sigma_x^2)}{\sigma_x^2(1+\sigma_x^2)} - \frac{x_1^2}{(1+\sigma_x^2)\sigma_x^2}\right)} \cdot 1$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}\left(\frac{x_1^2 + \sigma_x^2 x_1^2 - x_1^2}{(1+\sigma_x^2)\sigma_x^2}\right)} \cdot 1$$

$$= \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}\left(\frac{x_1^2}{1+\sigma_x^2}\right)} \cdot 1$$

Kalman Filters

$$F_1 = \hat{\alpha} N(x_1, \sigma_z^2)(z_1) e^{\frac{-1}{2}(\frac{x_1^2}{1+\sigma_x^2})}$$

$$= \hat{\alpha} e^{\frac{-1}{2}(\frac{(z_1 - x_1)^2}{\sigma_z^2})} e^{\frac{-1}{2}(\frac{x_1^2}{1+\sigma_x^2})}$$

$$= \hat{\alpha} e^{\frac{-1}{2}(\frac{(z_1 - x_1)^2}{\sigma_z^2} + \frac{x_1^2}{1+\sigma_x^2})}$$

$$= \hat{\alpha} e^{\frac{-1}{2}(\frac{(z_1 - x_1)^2(1+\sigma_x^2) + \sigma_z^2 x_1^2}{\sigma_z^2(1+\sigma_x^2)})}$$

$$ax^2 + bx + c = a(x - \frac{-b}{2a})^2 + (c - \frac{b^2}{4a})$$

$$= \hat{\alpha} e^{\frac{-1}{2}\left((\frac{1+\sigma_x^2+\sigma_z^2}{\sigma_z^2(1+\sigma_x^2)})x_1^2 + (-2z_1/\sigma_z^2)x_1 + z_1^2/\sigma_z^2\right)}$$

$$= \hat{\alpha} e^{\frac{-1}{2} a(x_1 - \frac{-b}{2a})^2 + (c - \frac{b^2}{4a})}$$

$$= \hat{\alpha} e^{\frac{-1}{2}(c - \frac{b^2}{4a})} e^{\frac{-1}{2} a(x_1 - \frac{-b}{2a})^2}$$

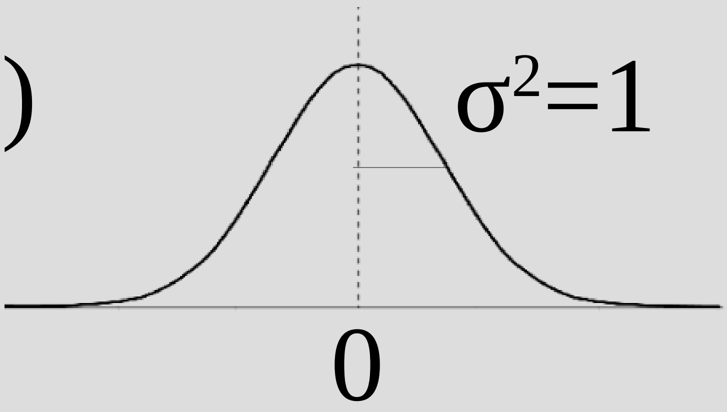
$$= \hat{\alpha}' e^{\frac{-1}{2} a(x_1 - \frac{-b}{2a})^2}$$

gross after plugging in a,b,c (see book)

Kalman: Frisbee in the Dark

$$a = \frac{1+\sigma_x^2+\sigma_z^2}{\sigma_z^2(1+\sigma_x^2)}, b = -2z_1/\sigma_z^2, c = z_1^2/\sigma_z^2$$

Initially your friend is N(0,1)



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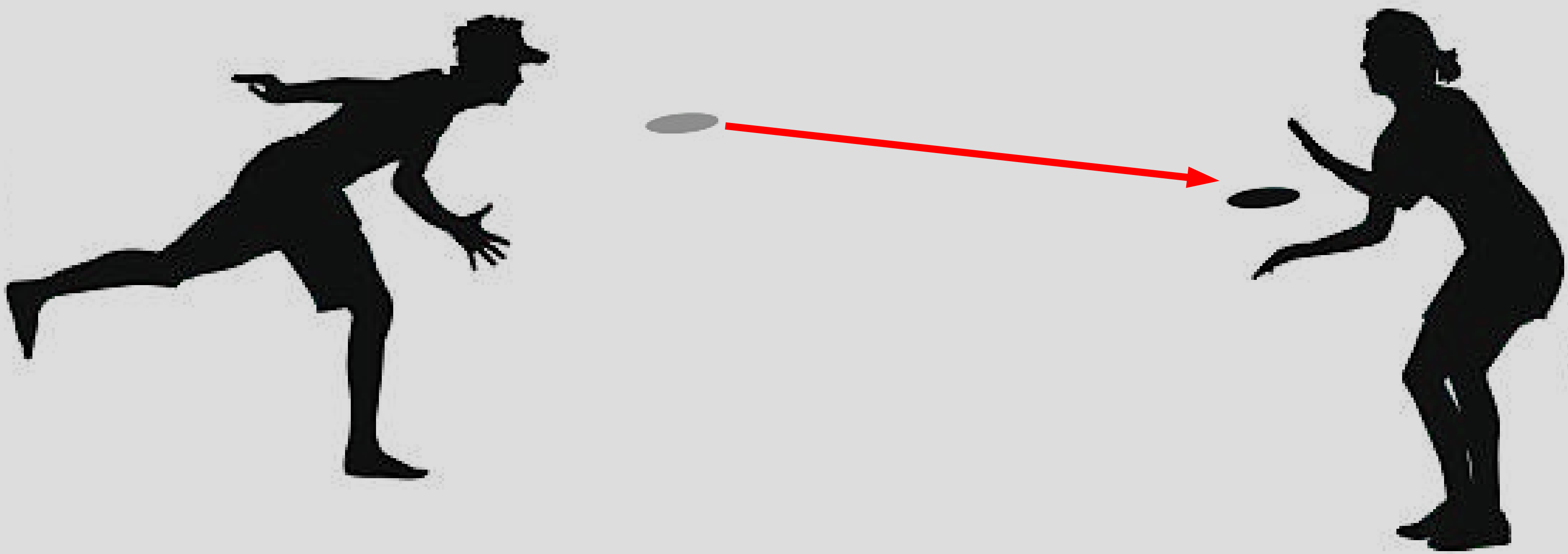
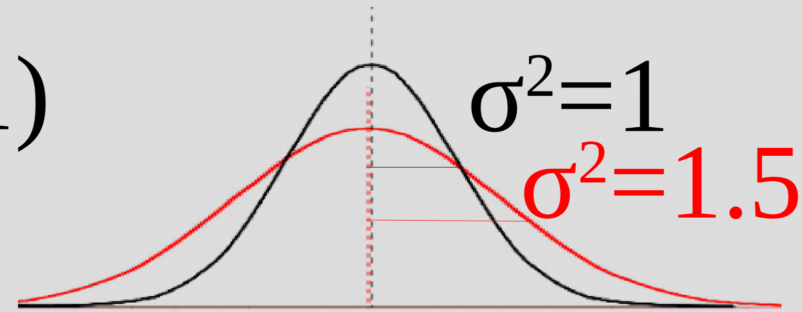
$a = \frac{1+\sigma_x^2+\sigma_z^2}{\sigma_z^2(1+\sigma_x^2)}, b = -2z_1/\sigma_z^2, c = z_1^2/\sigma_z^2$



Initially your friend is N(0,1)

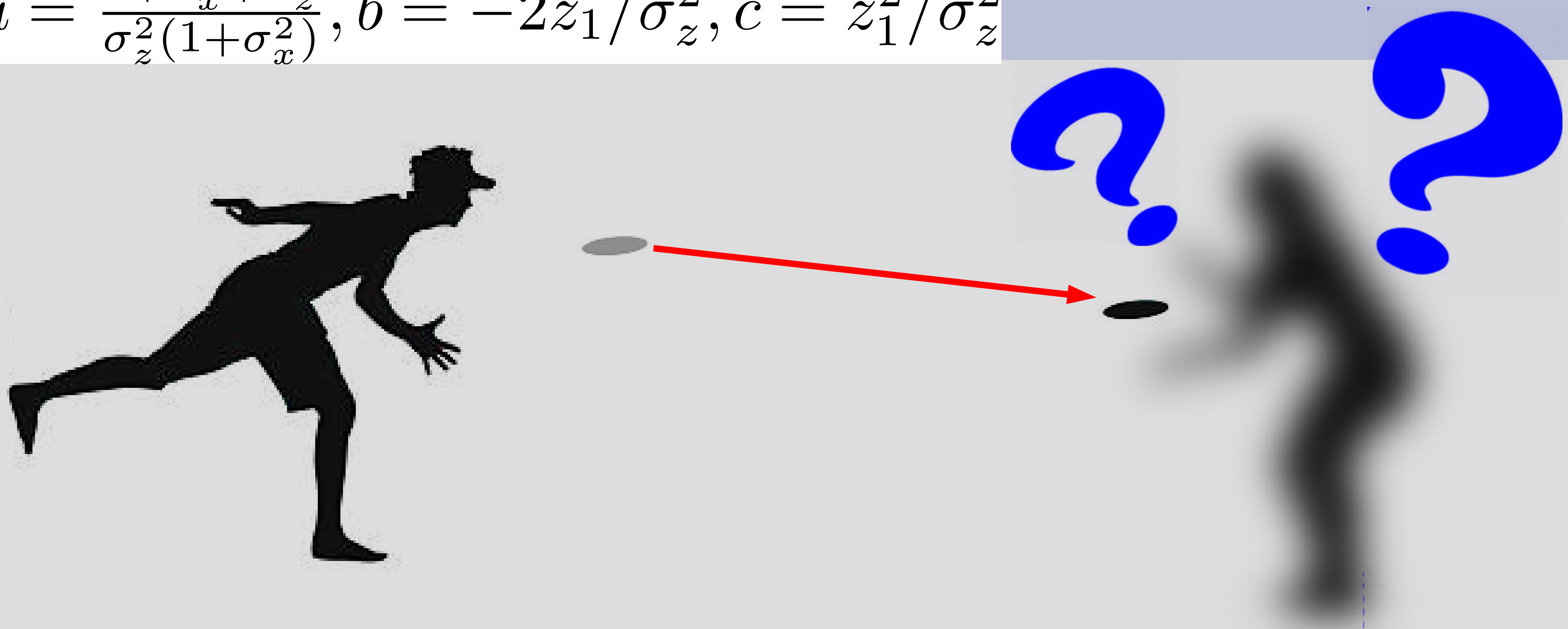
Throw not perfect, so friend has to move N(0,1.5)

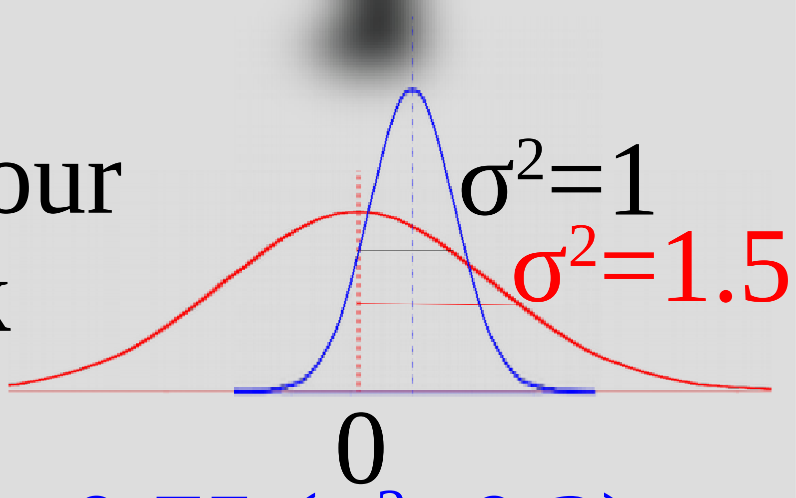


(i.e. move from black to red)

Kalman: Frisbee in the Dark

$$a = \frac{1+\sigma_x^2+\sigma_z^2}{\sigma_z^2(1+\sigma_x^2)}, b = -2z_1/\sigma_z^2, c = z_1^2/\sigma_z^2$$



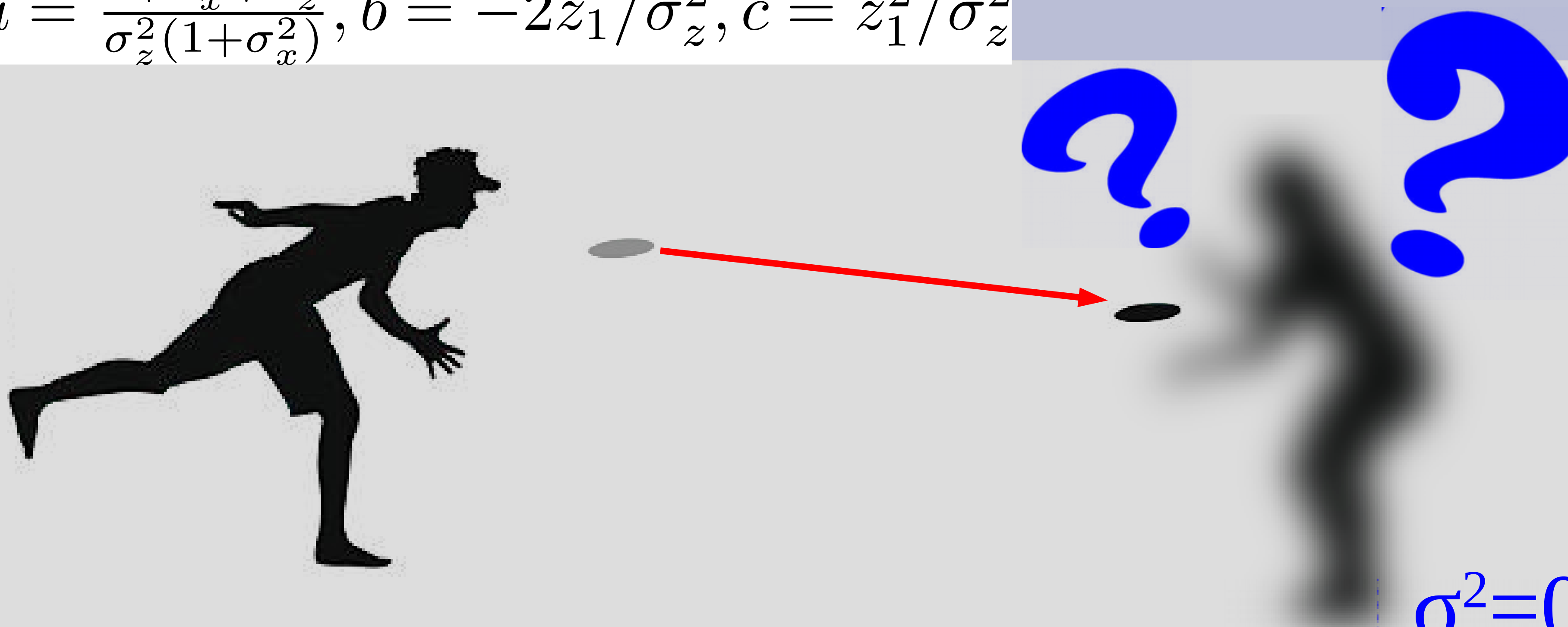


But you can't actually see your friend too clearly in the dark

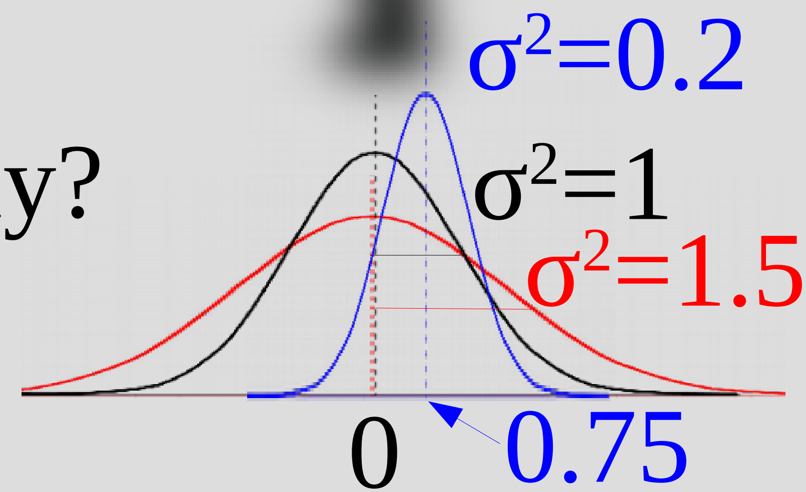
You thought you saw them at 0.75 (σ^2=0.2)

Kalman: Frisbee in the Dark

$$a = \frac{1+\sigma_x^2+\sigma_z^2}{\sigma_z^2(1+\sigma_x^2)}, b = -2z_1/\sigma_z^2, c = z_1^2/\sigma_z^2$$



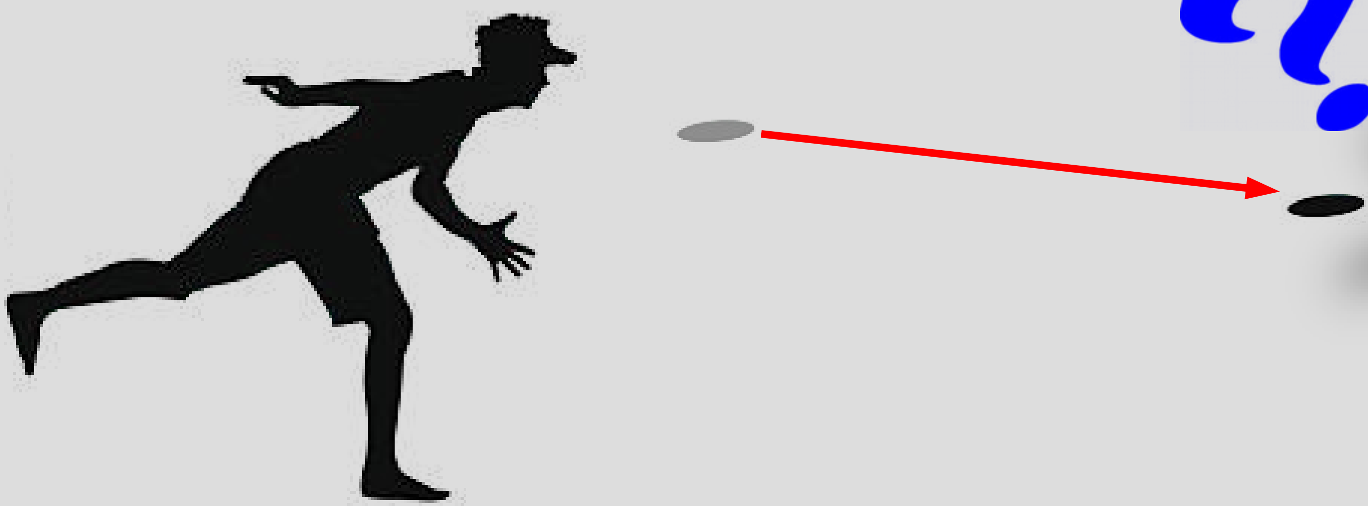
Where is your friend actually?

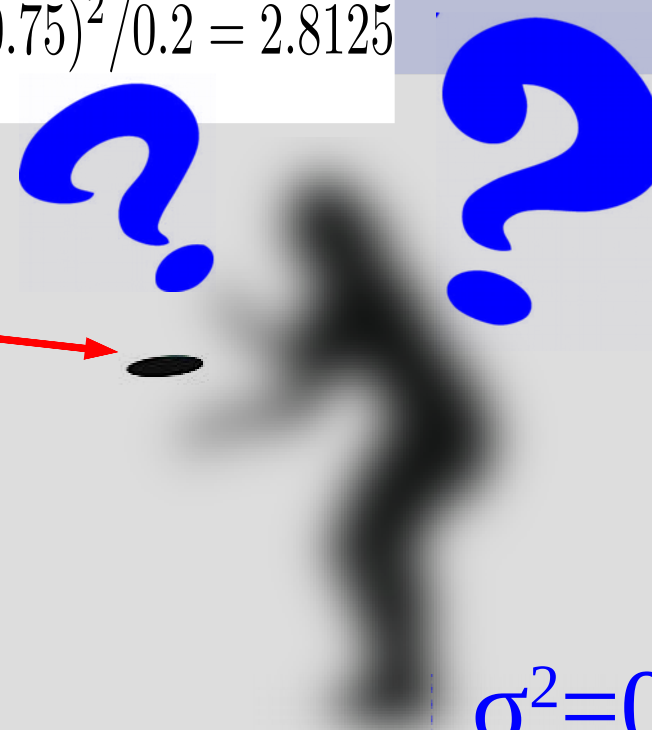


$$F_1 = \hat{\alpha}' e^{\frac{-1}{2} a (x_1 - \frac{-b}{2a})^2}$$

Kalman: Frisbee in the Dark

$$a = \frac{1 + 1.5 + 0.2}{0.2(1 + 1.5)} = 5.4, b = -2(0.75)/0.2 = -7.5, c = (0.75)^2/0.2 = 2.8125$$





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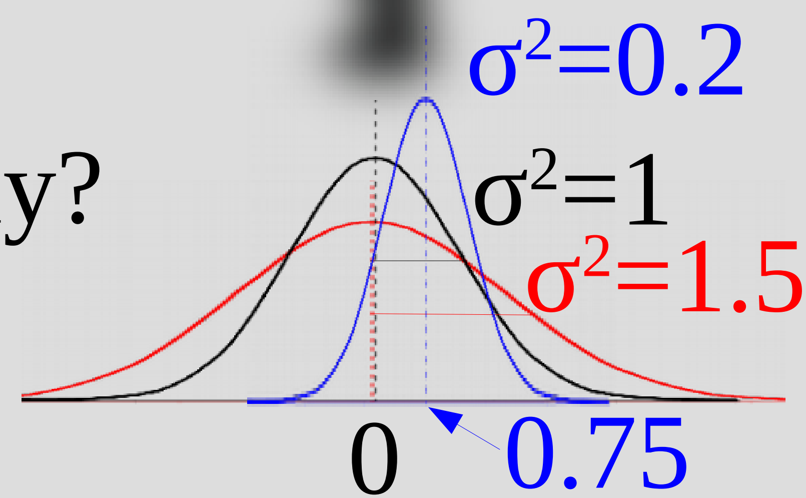

$$F_1 = \hat{\alpha}' e^{\frac{-1}{2} a (x_1 - \frac{-b}{2a})^2}$$
$$= N(\frac{-b}{2a}, \frac{1}{a})$$
$$= N(0.694, 0.185)$$

Probably 0.05 "left" of where you "saw" them

Kalman Filters

So the filtered "forward" message for throw 1 is: $N(0.694, 0.185)$

To find the filtered "forward" message for throw 2, use $N(0.694, 0.185)$ instead of $N(0, 1)$ (this does change the equations as you need to involve a μ for the old $N(0, 1)$)

The book gives you the full messy equations:

$$\mu_{t+1} = \frac{(\sigma_t^2+\sigma_x^2)z_{t+1}+\sigma_z^2\mu_t}{\sigma_t^2+\sigma_x^2+\sigma_z^2} \qquad \sigma_{t+1}^2 = \frac{(\sigma_t^2+\sigma_x^2)\sigma_z^2}{\sigma_t^2+\sigma_x^2+\sigma_z^2}$$

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Kalman Filters

The full Kalman filter is done with multiple numbers (matrices)

covariance matrix

Here a Gaussian is: $N(\mu, \Sigma) = \alpha e^{-\frac{1}{2}\left((x-\mu)^T \Sigma^{-1} (x-\mu)\right)}$

Bayes net is: (F and H are “linear” matrix)

$$P(x_{t+1} | x_t) = N(Fx_t, \Sigma_x)(x_{t+1})$$

$$P(z_t | x_t) = N(Hx_t, \Sigma_z)(z_t)$$

identity matrix

Then filter update is: $\Sigma_{t+1} = (I - K_{t+1}H)(F\Sigma_t F^T + \Sigma_x)$

$$\mu_{t+1} = F\mu_t + K_{t+1}(z_{t+1} - HF\mu_t)$$

yikes...

$$K_{t+1} = (F\Sigma_t F^T + \Sigma_x)H^T(H(F\Sigma_t F^T + \Sigma_x)H^T + \Sigma_z)^{-1}$$

Kalman Filters

Often we use $\begin{bmatrix} x \\ v_x \end{bmatrix}$ for a 1-dimensional problem with both position and velocity

To update x_{t+1}, we would want: $x_{t+1} = x_t + v_x$

In matrix form:

$$P(x_{t+1}|x_t) = N(Fx_t, \Sigma_x)(x_{t+1})$$

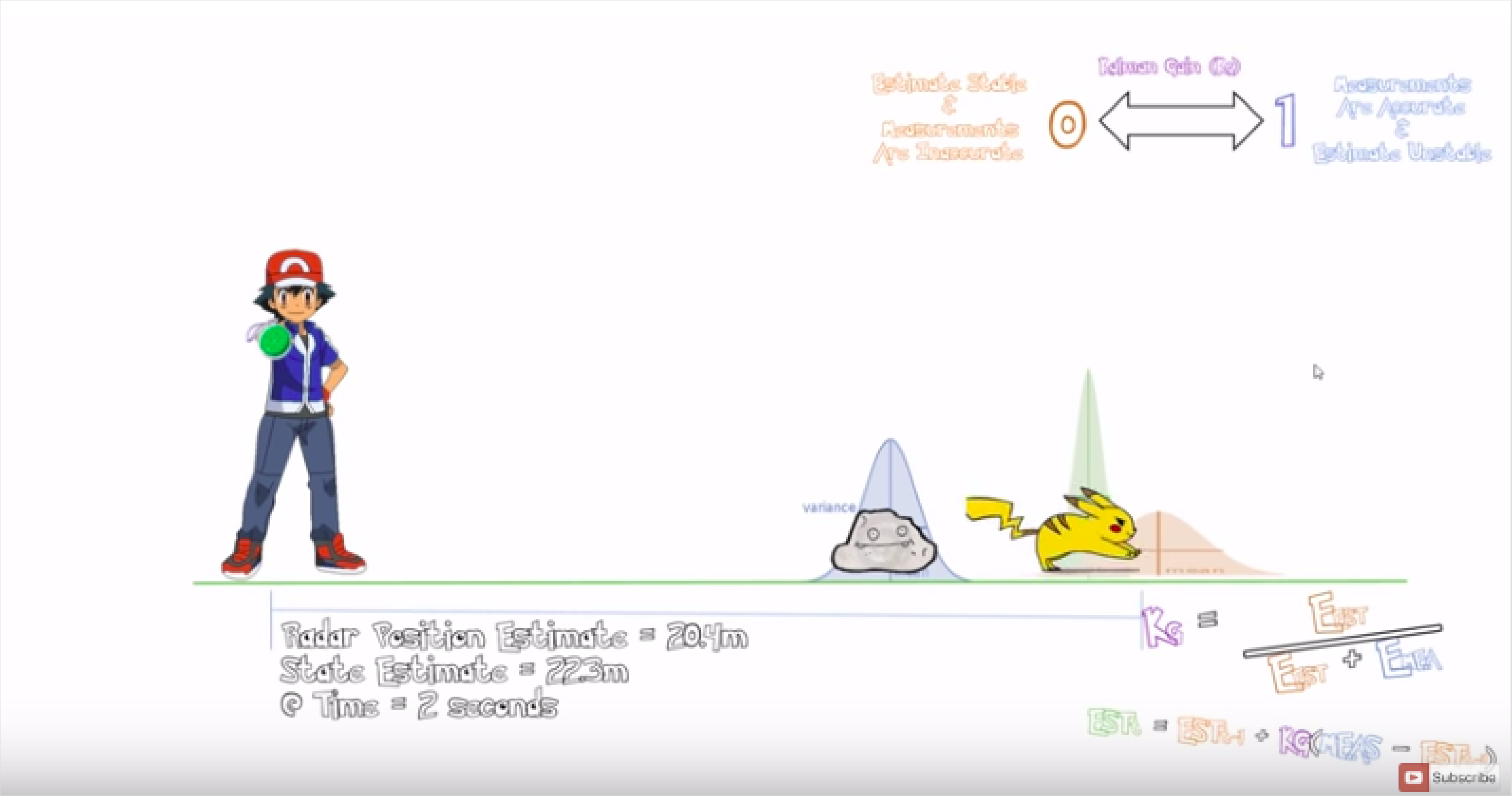
$F = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ so: $Fx_t = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ v_x \end{bmatrix} = \begin{bmatrix} x + v_x \\ v_x \end{bmatrix}$

So our "mean" at t+1 is [our position at $x+v_x$]

Kalman Filters

Here's a Pokemon example (not technical)

https://www.youtube.com/watch?v=bm3cwEP2nUo



Kalman Filters

Downsides?

In order to get “simple” equations, we are limited to the linear Gaussian assumption

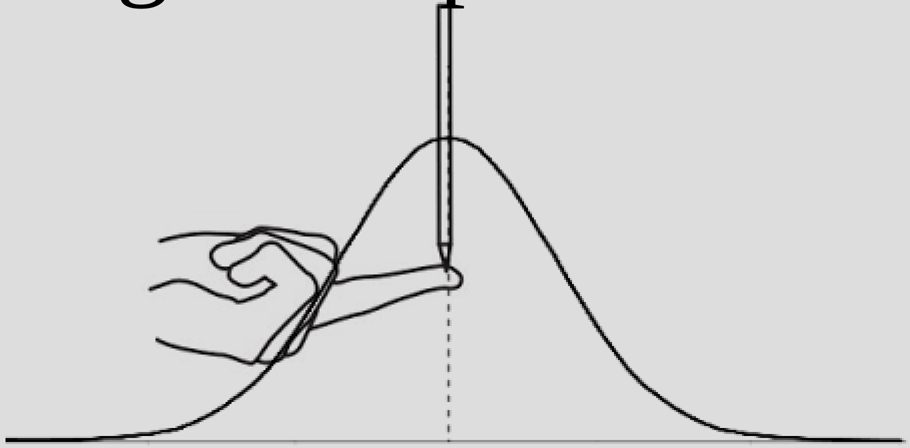
However, there are some cases when this assumption does not work very well at all

Kalman Filters

Consider the example of balancing a pencil on your finger

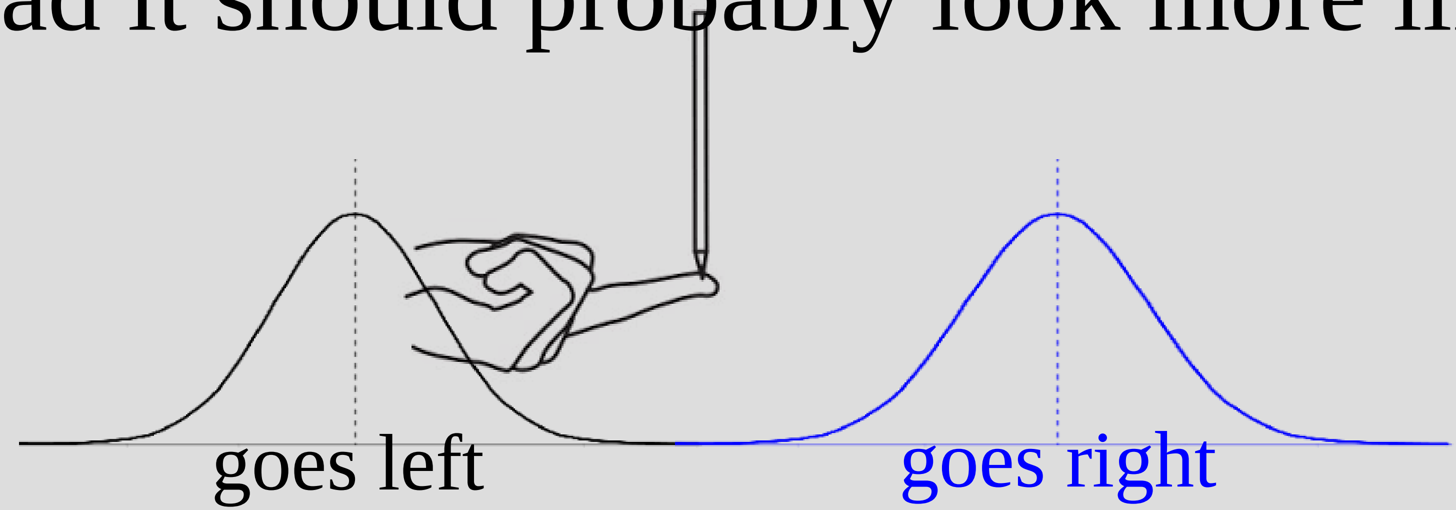
How far to the left/right will the pencil fall?

Below is not a good representation:



Kalman Filters

Instead it should probably look more like:



... where you are deciding between two options, but you are not sure which one

The Kalman filter can handle this as well (just keep 2 sets of equations and use more likely)

Kalman Filters

Unfortunately if you repeat this “pencil balance” on the new spot... you would need 4 sets of equations

3rd attempt: 8 equations
4th attempt: 16 equations
... this exponential amount of work/memory cannot be done for a large HMM