ERROR AND SENSITIVITY ANALYSIS FOR SYSTEMS OF LINEAR EQUATIONS

- **Conditioning of linear systems.**
- **Estimating errors for solutions of linear systems**
- **(Normwise) Backward error analysis**
- **Estimating condition numbers ..**

Perturbation analysis for linear systems ($Ax = b$)

Question addressed by perturbation analysis: determine the variation of the solution x when the data, namely A and b, undergoes small variations. Problem is Ill-conditioned if small variations in data cause very large variation in the solution.

Setting:

➤ We perturb A into $A + E$ and b into $b + e_b$. Can we bound the resulting change (perturbation) to the solution?

Preparation: We begin with a lemma for a simple case

Rigorous norm-based error bounds

LEMMA: If $\|E\| < 1$ then $I - E$ is nonsingular and

$$\|(I-E)^{-1}\| \le \frac{1}{1-\|E\|}$$

Proof is based on following 5 steps

a) Show: If $\|E\| < 1$ then $I - E$ is nonsingular

b) Show: $(I - E)(I + E + E^2 + \cdots + E^k) = I - E^{k+1}$.

c) From which we get:

$$(I-E)^{-1} = \sum_{i=0}^{k} E^i + (I-E)^{-1}E^{k+1} \rightarrow$$

d) $(I - E)^{-1} = \lim_{k\to\infty} \sum_{i=0}^{k} E^i$. We write this as

$$(I-E)^{-1} = \sum_{i=0}^{\infty} E^i$$

e) Finally:

$$\begin{aligned}\|(I-E)^{-1}\| &= \left\| \lim_{k\to\infty} \sum_{i=0}^{k} E^i \right\| = \lim_{k\to\infty} \left\| \sum_{i=0}^{k} E^i \right\| \\ &\le \lim_{k\to\infty} \sum_{i=0}^{k} \|E^i\| \le \lim_{k\to\infty} \sum_{i=0}^{k} \|E\|^i \\ &\le \frac{1}{1-\|E\|}\end{aligned}$$

➤ Can generalize result:

LEMMA: If A is nonsingular and $\|A^{-1}\| \; \|E\| < 1$ then $A + E$ is non-singular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\| \; \|E\|}$$

➤ Proof is based on relation $A + E = A(I + A^{-1}E)$ and use of previous lemma.

➤ Now we can prove the main theorem:

THEOREM 1: Assume that $(A + E)y = b + e_b$ and $Ax = b$ and that $\|A^{-1}\|\|E\| < 1$. Then $A + E$ is nonsingular and

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \; \|A\|}{1 - \|A^{-1}\| \; \|E\|}\left(\frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|}\right)$$

Proof: From $(A + E)y = b + e_b$ and $Ax = b$ we get $(A + E)(y - x) = e_b - Ex$. Hence:

$$y - x = (A + E)^{-1}(e_b - Ex)$$

Taking norms $\longrightarrow$ $\|y - x\| \leq \|(A + E)^{-1}\| \; [\|e_b\| + \|E\|\|x\|]$
Dividing by $\|x\|$ and using result of lemma

$$\begin{aligned}\frac{\|y - x\|}{\|x\|} &\leq \|(A + E)^{-1}\| \; [\|e_b\|/\|x\| + \|E\|] \\ &\leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}[\|e_b\|/\|x\| + \|E\|] \\ &\leq \frac{\|A^{-1}\|\|A\|}{1 - \|A^{-1}\|\|E\|}\left[\frac{\|e_b\|}{\|A\|\|x\|} + \frac{\|E\|}{\|A\|}\right]\end{aligned}$$

Result follows by using inequality $\|A\|\|x\| \geq \|b\|$.... QED

The quantity $\boxed{\kappa(A) = \|A\| \; \|A^{-1}\|}$ is called the condition number of the linear system with respect to the norm $\|.\|$. When using the p-norms we write:

$$\kappa_p(A) = \|A\|_p \|A^{-1}\|_p$$

➤ Note: $\kappa_2(A) = \sigma_{max}(A)/\sigma_{min}(A)$ = ratio of largest to smallest singular values of A. Allows to define $\kappa_2(A)$ when A is not square.

➤ Determinant *is not* a good indication of sensitivity

➤ Small eigenvalues *do not* always give a good indication of poor conditioning.

Example: Consider, for a large α, the $n \times n$ matrix

$$A = I + \alpha e_1 e_n^T$$

➤ Inverse of A is : $A^{-1} = I - \alpha e_1 e_n^T$ ➤ For the ∞-norm we have

$$\|A\|_\infty = \|A^{-1}\|_\infty = 1 + |\alpha|$$

so that $\kappa_\infty(A) = (1 + |\alpha|)^2.$

➤ Can give a very large condition number for a large α – but all the eigenvalues of A are equal to one.

✍1 Show that $\kappa(I) = 1$;

✍2 Show that $\kappa(A) \geq 1$;

✍3 Show that $\kappa(A) = \kappa(A^{-1})$

✍4 Show that for $\alpha \neq 0$, we have $\kappa(\alpha A) = \kappa(A)$

Simplification when $e_b = 0$:

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \ \|E\|}{1 - \|A^{-1}\| \ \|E\|}$$

Simplification when $E = 0$:

$$\frac{\|x - y\|}{\|x\|} \leq \|A^{-1}\| \ \|A\| \frac{\|e_b\|}{\|b\|}$$

➤ Slightly less general form: Assume that $\|E\|/\|A\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ and $\delta\kappa(A) < 1$ then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}$$

✍5 Show the above result

Another common form:

THEOREM 2: Let $(A + \Delta A)y = b + \Delta b$ and $Ax = b$ where $\|\Delta A\| \leq \epsilon\|E\|$, $\|\Delta b\| \leq \epsilon\|e_b\|$, and assume that $\epsilon\|A^{-1}\|\|E\| < 1$. Then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\epsilon \ \|A^{-1}\| \ \|A\|}{1 - \epsilon\|A^{-1}\| \ \|E\|} \left(\frac{\|e_b\|}{\|b\|} + \frac{\|E\|}{\|A\|} \right)$$

➤ Results to be seen later are of this type.

Normwise backward error

➤ We solve $Ax = b$ and find an approximate solution y

Question: Find smallest perturbation to apply to A, b so that *exact* solution of perturbed system is y

Normwise backward error in just A or b

Suppose we model entire perturbation in RHS $\boldsymbol{b}$.

- Let $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{y}$ be the residual. Then $\boldsymbol{y}$ satisfies $\boldsymbol{A}\boldsymbol{y} = \boldsymbol{b} + \Delta\boldsymbol{b}$ with $\Delta\boldsymbol{b} = -\boldsymbol{r}$ exactly.
- The relative perturbation to the RHS is $\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|}$.

Suppose we model entire perturbation in matrix $\boldsymbol{A}$.

- Then $\boldsymbol{y}$ satisfies $\left(\boldsymbol{A} + \frac{\boldsymbol{r}\boldsymbol{y}^T}{\boldsymbol{y}^T\boldsymbol{y}}\right)\boldsymbol{y} = \boldsymbol{b}$
- The relative perturbation to the matrix is

$$\left\|\frac{\boldsymbol{r}\boldsymbol{y}^T}{\boldsymbol{y}^T\boldsymbol{y}}\right\|_2 / \|\boldsymbol{A}\|_2 = \frac{\|\boldsymbol{r}\|_2}{\|\boldsymbol{A}\|\|\boldsymbol{y}\|_2}$$

Normwise backward error in both A & b

For a given $\boldsymbol{y}$ and given perturbation directions $\boldsymbol{E}, \boldsymbol{e}_b$, we define the Normwise backward error:

$$\eta_{E,e_b}(\boldsymbol{y}) = \min\{\epsilon \mid (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{y} = \boldsymbol{b} + \Delta\boldsymbol{b}; \text{ where } \Delta\boldsymbol{A}, \Delta\boldsymbol{b} \text{ satisfy: } \|\Delta\boldsymbol{A}\| \le \epsilon\|\boldsymbol{E}\|; \text{ and } \|\Delta\boldsymbol{b}\| \le \epsilon\|\boldsymbol{e}_b\|\}$$

In other words $\eta_{E,e_b}(\boldsymbol{y})$ is the smallest ϵ for which

$$(1)\begin{cases} (\boldsymbol{A} + \Delta\boldsymbol{A})\boldsymbol{y} = & \boldsymbol{b} + \Delta\boldsymbol{b}; \\ \|\Delta\boldsymbol{A}\| \le \epsilon\|\boldsymbol{E}\|; & \|\Delta\boldsymbol{b}\| \le \epsilon\|\boldsymbol{e}_b\| \end{cases}$$

- $\boldsymbol{y}$ is given (a computed solution). $\boldsymbol{E}$ and $\boldsymbol{e}_b$ to be selected (most likely 'directions of perturbation for $\boldsymbol{A}$ and $\boldsymbol{b}$').
- Typical choice: $\boldsymbol{E} = \boldsymbol{A}$, $\boldsymbol{e}_b = \boldsymbol{b}$

✍6 Explain why this is not unreasonable

Let $\boldsymbol{r} = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{y}$. Then we have:

THEOREM 3: $\eta_{E,e_b}(\boldsymbol{y}) = \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{E}\|\|\boldsymbol{y}\| + \|\boldsymbol{e}_b\|}$

Normwise backward error is for case $\boldsymbol{E} = \boldsymbol{A}, \boldsymbol{e}_b = \boldsymbol{b}$:

$$\eta_{A,b}(\boldsymbol{y}) = \frac{\|\boldsymbol{r}\|}{\|\boldsymbol{A}\|\|\boldsymbol{y}\| + \|\boldsymbol{b}\|}$$

✍7 Show how this can be used in practice as a means to stop some iterative method which computes a sequence of approximate solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

✍8 Consider the 6×6 Vandermonde system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ where $a_{ij} = j^{2(i-1)}$, $\boldsymbol{b} = \boldsymbol{A} * [1, 1, \cdots, 1]^T$. We perturb $\boldsymbol{A}$ by $\boldsymbol{E}$, with $|\boldsymbol{E}| \le 10^{-10}|\boldsymbol{A}|$ and $\boldsymbol{b}$ similarly and solve the system. Evaluate the backward error for this case. Evaluate the forward bound provided by Theorem 2. Comment on the results.

Estimating condition numbers.

- Often we just want to get a lower bound for condition number [it is 'worse than ...']
- We want to estimate $\|A\|\ \|A^{-1}\|$.
- The norm $\|A\|$ is usually easy to compute but $\|A^{-1}\|$ is not.
- We want: Avoid the expense of computing A^{-1} explicitly.

Idea:

- Select a vector v so that $\|v\| = 1$ but $\|Av\| = \tau$ is small.
- Then: $\|A^{-1}\| \geq 1/\tau$ (show why) and:

$$\kappa(A) \geq \frac{\|A\|}{\tau}$$

- Condition number worse than $\|A\|/\tau$.
- Typical choice for v: choose $[\cdots \pm 1 \cdots]$ with signs chosen on the fly during back-substitution to maximize the next entry in the solution, based on the upper triangular factor from Gaussian Elimination.
- Similar techniques used to estimate condition numbers of large matrices in matlab.

Condition numbers and near-singularity

- $1/\kappa \approx$ relative distance to nearest singular matrix.

Let A, B be two $n \times n$ matrices with A nonsingular and B singular. Then

$$\frac{1}{\kappa(A)} \leq \frac{\|A - B\|}{\|A\|}$$

Proof: B singular $\rightarrow\ \exists\ x \neq 0$ such that $Bx = 0$.

$$\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\|\ \|Ax\| = \|A^{-1}\|\|(A-B)x\| \\ \leq \|A^{-1}\|\ \|A - B\|\|x\|$$

Divide both sides by $\|x\| \times \kappa(A) = \|x\|\|A\|\ \|A^{-1}\|$ ➤ result. QED.

Example:

$$\text{let } A = \begin{pmatrix} 1 & 1 \\ 1 & 0.99 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Then $\frac{1}{\kappa_1(A)} \leq \frac{0.01}{2}$ ➤ $\kappa_1(A) \geq \frac{2}{0.01} = 200.$

- It can be shown that (Kahan)

$$\frac{1}{\kappa(A)} = \min_B \left\{ \frac{\|A - B\|}{\|A\|} \quad \middle| \quad \det(B) = 0 \right\}$$

Estimating errors from residual norms

Let $\tilde{x}$ an approximate solution to system $Ax = b$ (e.g., computed from an iterative process). We can compute the residual norm:

$$\|r\| = \|b - A\tilde{x}\|$$

Question: How to estimate the error $\|x - \tilde{x}\|$ from $\|r\|$?

➤ One option is to use the inequality

$$\frac{\|x-\tilde{x}\|}{\|x\|} \le \kappa(A) \, \frac{\|r\|}{\|b\|}.$$

➤ We must have an estimate of $\kappa(A)$.

Proof of inequality.

First, note that $A(x - \tilde{x}) = b - A\tilde{x} = r$. So:

$$\|x - \tilde{x}\| = \|A^{-1}r\| \le \|A^{-1}\| \, \|r\|$$

Also note that from the relation $b = Ax$, we get

$$\|b\| = \|Ax\| \le \|A\| \, \|x\| \quad \to \quad \|x\| \ge \frac{\|b\|}{\|A\|}$$

Therefore,

$$\frac{\|x - \tilde{x}\|}{\|x\|} \le \frac{\|A^{-1}\| \, \|r\|}{\|b\|/\|A\|} = \kappa(A)\frac{\|r\|}{\|b\|} \qquad \square$$

✍9 Show that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \ge \frac{1}{\kappa(A)} \, \frac{\|r\|}{\|b\|}.$$