The QR algorithm

➤ The most common method for solving small (dense) eigenvalue problems. The basic algorithm:

QR without shifts

1. Until Convergence Do:
2. Compute the QR factorization $A = QR$
3. Set $A := RQ$
4. EndDo

➤ "Until Convergence" means "Until A becomes close enough to an upper triangular matrix"

➤ Note: $A_{new} = RQ = Q^H(QR)Q = Q^H AQ$

➤ A_{new} *Unitarily similar to* A $\rightarrow$ Spectrum does not change

➤ Convergence analysis complicated – but insight: we are implicitly doing a QR factorization of A^k:

	QR-Factorize:	Multiply backward:	
Step 1	$A_0 = Q_0R_0$	$A_1 = R_0Q_0$	
Step 2	$A_1 = Q_1R_1$	$A_2 = R_1Q_1$	
Step 3:	$A_2 = Q_2R_2$	$A_3 = R_2Q_2$	Then:

$$\begin{aligned}[Q_0Q_1Q_2][R_2R_1R_0] &= Q_0Q_1A_2R_1R_0\\ &= Q_0(Q_1R_1)(Q_1R_1)R_0\\ &= Q_0A_1A_1R_0, \qquad A_1 = R_0Q_0 \rightarrow\\ &= \underbrace{(Q_0R_0)}_{A}\ \underbrace{(Q_0R_0)}_{A}\ \underbrace{(Q_0R_0)}_{A} = A^3\end{aligned}$$

➤ $[Q_0Q_1Q_2][R_2R_1R_0]$ == QR factorization of A^3

➤ This helps analyze the algorithm (details skipped)

➤ Above basic algorithm is never used as is in practice. Two variations:

(1) Use shift of origin and

(2) Start by transforming A into an Hessenberg matrix

Practical QR algorithms: Shifts of origin

Observation: (from theory): Last row converges fastest. Convergence is dictated by $\frac{|\lambda_n|}{|\lambda_{n-1}|}$

➤ We will now consider only the real symmetric case.

➤ Eigenvalues are real.

➤ $A^{(k)}$ remains symmetric throughout process.

➤ As k goes to infinity the last column and row (except $a_{nn}^{(k)}$) converge to zero quickly.,,

➤ and $a_{nn}^{(k)}$ converges to lowest eigenvalue.

$$A^{(k)} = \left(\begin{array}{ccccc|c} . & . & . & . & . & a \\ . & . & . & . & . & a \\ . & . & . & . & . & a \\ . & . & . & . & . & a \\ . & . & . & . & . & a \\ \hline a & a & a & a & a & a \end{array}\right)$$

➤ Idea: Apply QR algorithm to $A^{(k)} - \mu I$ with $\mu = a_{nn}^{(k)}$. Note: eigenvalues of $A^{(k)} - \mu I$ are shifted by μ, and eigenvectors are the same.

QR with shifts

1. Until row $a_{in}, 1 \leq i < n$ converges to zero DO:
2. Obtain next shift (e.g. $\mu = a_{nn}$)
3. $A - \mu I = QR$
5. Set $A := RQ + \mu I$
6. EndDo

➤ Convergence (of last row) is cubic at the limit! [for symmetric case]

➤ Result of algorithm:

$$A^{(k)} = \left(\begin{array}{ccccc|c} . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ \hline 0 & 0 & 0 & 0 & 0 & \lambda_n \end{array}\right)$$

➤ Next step: deflate, i.e., apply above algorithm to $(n-1) \times (n-1)$ upper block.

Practical algorithm: Use the Hessenberg Form

Recall: Upper Hessenberg matrix is such that

$$a_{ij} = 0 \text{ for } j < i - 1$$

Observation: The QR algorithm preserves Hessenberg form (tridiagonal form in symmetric case). Results in substantial savings.

Transformation to Hessenberg form

$$\begin{pmatrix} \star & \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star & \star \end{pmatrix}$$

➤ Want $H_1 A H_1^T = H_1 A H_1$ to have the form shown on the right

➤ Consider the first step only on a 6×6 matrix

- Choose a w in $H_1 = I - 2ww^T$ to make the first column have zeros from position 3 to n. So $w_1 = 0$.
- Apply to left: $B = H_1 A$
- Apply to right: $A_1 = BH_1$.

Main observation: the Householder matrix H_1 which transforms the column $A(2:n,1)$ into e_1 works only on rows 2 to n. When applying the transpose H_1 to the right of $B = H_1 A$, we observe that only columns 2 to n will be altered. So the first column will retain the desired pattern (zeros below row 2).

- Algorithm continues the same way for columns 2, ...,$n-2$.

QR for Hessenberg matrices

- Need the "Implicit Q theorem"

Suppose that $Q^T AQ$ is an unreduced upper Hessenberg matrix. Then columns 2 to n of Q are determined uniquely (up to signs) by the first column of Q.

- In other words if $V^T AV = G$ and $Q^T AQ = H$ are both Hessenberg and $V(:,1) = Q(:,1)$ then $V(:,i) = \pm Q(:,i)$ for $i = 2:n$.

Implication: To compute $A_{i+1} = Q_i^T AQ_i$ we can:

- Compute 1st column of Q_i [== scalar $\times A(:,1)$]
- Choose other columns so Q_i = unitary, and A_{i+1} = Hessenberg.

- W'll do this with Givens rotations:

Example: With $n = 5$:

$$A = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

1. Choose $G_1 = G(1,2,\theta_1)$ *so that* $(G_1^T A_0)_{21} = 0$

- $$A_1 = G_1^T AG_1 = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ + & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

2. Choose $G_2 = G(2,3,\theta_2)$ *so that* $(G_2^T A_1)_{31} = 0$

- $$A_2 = G_2^T A_1 G_2 = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & + & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

3. Choose $G_3 = G(3,4,\theta_3)$ *so that* $(G_3^T A_2)_{42} = 0$

- $$A_3 = G_3^T A_2 G_3 = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & + & * & * \end{pmatrix}$$

4. *Choose* $G_4 = G(4,5,\theta_4)$ *so that* $(G_4^T A_3)_{53} = 0$

➤ $A_4 = G_4^T A_3 G_4 = \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$

➤ Process known as "Bulge chasing"

➤ Similar idea for the symmetric (tridiagonal) case

The symmetric eigenvalue problem: Basic facts

➤ Consider the Schur form of a real symmetric matrix A:

$$A = QRQ^H$$

Since $A^H = A$ then $R = R^H$ ➤

Eigenvalues of A are real

and

There is an orthonormal basis of eigenvectors of A

In addition, Q can be taken to be real when A is real.

$(A - \lambda I)(u + iv) = 0 \rightarrow (A - \lambda I)u = 0$ & $(A - \lambda I)v = 0$

➤ Can select eigenvector to be either u or v

The min-max theorem (Courant-Fischer)

Label eigenvalues decreasingly:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$$

The eigenvalues of a Hermitian matrix A are characterized by the relation

$$\lambda_k = \max_{S,\ \dim(S)=k} \quad \min_{x\in S, x\neq 0} \frac{(Ax,x)}{(x,x)}$$

Proof: Preparation: Since A is symmetric real (or Hermitian complex) there is an orthonormal basis of eigenvectors $u_1, u_2, \cdots, u_n$. Express any vector x in this basis as $x = \sum_{i=1}^n \alpha_i u_i$. Then : $(Ax,x)/(x,x) = [\sum \lambda_i |\alpha_i|^2]/[\sum |\alpha_i|^2]$.

(a) Let S be any subspace of dimension k and let $\mathcal{W} = \text{span}\{u_k, u_{k+1}, \cdots, u_n\}$. A dimension argument (used before) shows that $S \cap \mathcal{W} \neq \{0\}$. So there is a

non-zero x_w in $S \cap \mathcal{W}$. Express this x_w in the eigenbasis as $x_w = \sum_{i=k}^n \alpha_i u_i$. Then since $\lambda_i \leq \lambda_k$ for $i \geq k$ we have:

$$\frac{(Ax_w, x_w)}{(x_w, x_w)} = \frac{\sum_{i=k}^n \lambda_i |\alpha_i|^2}{\sum_{i=k}^n |\alpha_i|^2} \leq \lambda_k$$

So for any subspace S of dim. k we have $\min_{x\in S, x\neq 0}(Ax,x)/(x,x) \leq \lambda_k$.

(b) We now take $S_* = \text{span}\{u_1, u_2, \cdots, u_k\}$. Since $\lambda_i \geq \lambda_k$ for $i \leq k$, for this particular subspace we have:

$$\min_{x\,\in\,S_*,\ x\neq 0} \frac{(Ax,x)}{(x,x)} = \min_{x\,\in\,S_*,\ x\neq 0} \frac{\sum_{i=1}^k \lambda_i |\alpha_i|^2}{\sum_{i=k}^n |\alpha_i|^2} = \lambda_k.$$

(c) The results of (a) and (b) imply that the max over all subspaces S of dim. k of $\min_{x\in S, x\neq 0}(Ax,x)/(x,x)$ is equal to λ_k $\square$

➤ Consequences:

$$\lambda_1 = \max_{x\neq 0} \frac{(Ax, x)}{(x, x)} \qquad \lambda_n = \min_{x\neq 0} \frac{(Ax, x)}{(x, x)}$$

➤ Actually 4 versions of the same theorem. 2nd version:

$$\lambda_k = \min_{S,\ \dim(S)=n-k+1} \quad \max_{x\in S, x\neq 0} \frac{(Ax, x)}{(x, x)}$$

➤ Other 2 versions come from ordering eigenvalues increasingly instead of decreasingly.

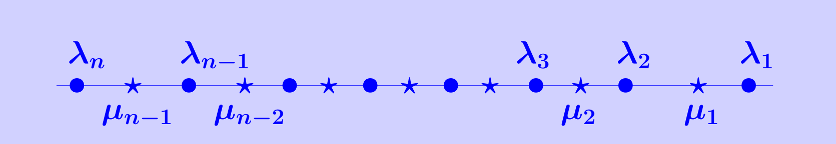
✍1 Write down all 4 versions of the theorem

✍2 Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of A.

➤ Interlacing Theorem: Denote the $k \times k$ principal submatrix of A as A_k, with eigenvalues $\{\lambda_i^{[k]}\}_{i=1}^{k}$. Then

$$\lambda_1^{[k]} \geq \lambda_1^{[k-1]} \geq \lambda_2^{[k]} \geq \lambda_2^{[k-1]} \geq \cdots \lambda_{k-1}^{[k-1]} \geq \lambda_k^{[k]}$$

Example: λ_i's = eigenvalues of A, μ_i's = eigenvalues of A_{n-1}:



➤ Many uses.

➤ For example: interlacing theorem for roots of orthogonal polynomials

The Law of inertia (real symmetric matrices)

➤ Inertia of a matrix = [m, z, p] with m = number of < 0 eigenvalues, z = number of zero eigenvalues, and p = number of > 0 eigenvalues.

Sylvester's Law of inertia:	*If $X \in \mathbb{R}^{n\times n}$ is nonsingular, then A and X^TAX have the same inertia.*

✍3 Suppose that $A = LDL^T$ where L is unit lower triangular, and D diagonal. How many negative eigenvalues does A have?

✍4 Assume that A is tridiagonal. How many operations are required to determine the number of negative eigenvalues of A?

✍5 Devise an algorithm based on the inertia theorem to compute the i-th eigenvalue of a tridiagonal matrix.

✍6 Let $F \in \mathbb{R}^{m\times n}$, with $n < m$, and F of rank n.
What is the inertia of the matrix on the right: $\begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix}$
[Hint: use a block LU factorization]

➤ Note 1: Converse result also true: If A and B have same inertia they are congruent. [This part is easy to show]

➤ Note 2: result also true for Hermitian matrices (X^HAX has same inertia as A).

Bisection algorithm for tridiagonal matrices:

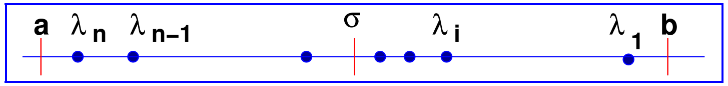
- Goal: to compute i-th eigenvalue of A (tridiagonal)

- Get interval $[a, b]$ containing spectrum [Gershgorin]: $a \le \lambda_n \le \cdots \le \lambda_1 \le b$

- Let $\sigma = (a + b)/2$ = middle of interval

- Calculate p = number of positive eigenvalues of $A - \sigma I$

  - If $p \ge i$ then $\lambda_i \in \ (\sigma,\ b] \rightarrow$ set $a := \sigma$



  - Else then $\lambda_i \in \ [a,\ \sigma] \rightarrow$ set $b := \sigma$

- Repeat until $b - a$ is small enough.

The QR algorithm for symmetric matrices

- Most important method used : reduce to tridiagonal form and apply the QR algorithm with shifts.

- Householder transformation to Hessenberg form yields a tridiagonal matrix because

$$HAH^T = A_1$$

is symmetric and also of Hessenberg form ➤ it is tridiagonal symmetric.

Tridiagonal form preserved by QR similarity transformation

Practical method

- How to implement the QR algorithm with shifts?

- It is best to use Givens rotations – can do a shifted QR step without explicitly shifting the matrix..

- Two most popular shifts:

$s = a_{nn}$ and s = smallest e.v. of $A(n-1:n, n-1:n)$

Jacobi iteration - Symmetric matrices

- Main idea: Rotation matrices of the form

$$J(p,q,\theta) = \begin{array}{c} \begin{pmatrix} 1 & \cdots & 0 & & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & & \cdots & & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ p \\ \\ q \\ \\ \\ \end{array}$$

$c = \cos\theta$ and $s = \sin\theta$ are so that $J(p,q,\theta)^T A J(p,q,\theta)$ has a zero in position (p,q) (and also (q,p))

- Frobenius norm of matrix is preserved – but diagonal elements become larger ➤ convergence to a diagonal.

- Let $B = J^T A J$ (where $J \equiv J_{p,q,\theta}$).
- Look at 2×2 matrix $B([p,q],[p,q])$ (matlab notation)
- Keep in mind that $a_{pq} = a_{qp}$ and $b_{pq} = b_{qp}$

$$\begin{pmatrix} b_{pp} & b_{pq} \\ b_{qp} & b_{qq} \end{pmatrix} = \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \begin{pmatrix} a_{pp} & a_{pq} \\ a_{qp} & a_{qq} \end{pmatrix} \begin{pmatrix} c & s \\ -s & c \end{pmatrix}$$
$$= \begin{pmatrix} c & -s \\ s & c \end{pmatrix} \left[\begin{array}{c|c} ca_{pp} - sa_{pq} & sa_{pp} + ca_{pq} \\ \hline ca_{qp} - sa_{qq} & sa_{pq} + ca_{qq} \end{array}\right]$$
$$= \left[\begin{array}{c|c} c^2a_{pp} + s^2a_{qq} - 2sc\, a_{pq} & (c^2 - s^2)a_{pq} - sc(a_{qq} - a_{pp}) \\ \hline * & c^2a_{qq} + s^2a_{pp} + 2sc\, a_{pq} \end{array}\right]$$

- Want: $$(c^2 - s^2)a_{pq} - sc(a_{qq} - a_{pp}) = 0$$

$$\frac{c^2 - s^2}{2sc} = \frac{a_{qq} - a_{pp}}{2a_{pq}} \equiv \tau$$

- Letting $t = s/c$ $(= \tan\theta)$ $\rightarrow$ quad. equation

$$t^2 + 2\tau t - 1 = 0$$

- $t = -\tau \pm \sqrt{1+\tau^2} = \frac{1}{\tau \pm \sqrt{1+\tau^2}}$
- Select sign to get a smaller t so $\theta \le \pi/4$.
- Then : $$c = \frac{1}{\sqrt{1+t^2}}; \qquad s = c * t$$
- Implemented in matlab script `jacrot(A,p,q)` –

- Define: $A_O = A - \mathrm{Diag}(A)$ $\equiv$ A 'with its diagonal entries replaced by zeros'
- Observations: (1) Unitary transformations preserve $\|.\|_F$. (2) Only changes are in rows and columns p and q.
- Let $B = J^T A J$ (where $J \equiv J_{p,q,\theta}$). Then,

$$a_{pp}^2 + a_{qq}^2 + 2a_{pq}^2 = b_{pp}^2 + b_{qq}^2 + 2b_{pq}^2 = b_{pp}^2 + b_{qq}^2$$

because $b_{pq} = 0$. Then, a little calculation leads to:

$$\begin{aligned} \|B_O\|_F^2 &= \|B\|_F^2 - \sum b_{ii}^2 = \|A\|_F^2 - \sum b_{ii}^2 \\ &= \|A\|_F^2 - \sum a_{ii}^2 + \sum a_{ii}^2 - \sum b_{ii}^2 \\ &= \|A_O\|_F^2 + (a_{pp}^2 + a_{qq}^2 - b_{pp}^2 - b_{qq}^2) \\ &= \|A_O\|_F^2 - 2a_{pq}^2 \end{aligned}$$

- $\|A_O\|_F$ will decrease from one step to the next.

✍7 Let $\|A_O\|_I = \max_{i \ne j} |a_{ij}|$. Show that

$$\|A_O\|_F \le \sqrt{n(n-1)}\|A_O\|_I$$

✍8 Use this to show convergence in the case when largest entry is zeroed at each step.