✍2 Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of $A$.

**Solution:** This comes from the fact that:

$$\begin{aligned}\|A\|_2^2 &= \max_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} \\ &= \max_{x\neq 0} \frac{(Ax, Ax)}{(x, x)} \\ &= \max_{x\neq 0} \frac{(A^TAx, A)}{(x, x)} \\ &= \lambda_{max}(A^TA) \\ &= \sigma_1^2\end{aligned}$$

$\square$

✍3 Suppose that $A = LDL^T$ where $L$ is unit lower triangular, and $D$ diagonal. How many negative eigenvalues does $A$ have?

**Solution:** It has as many negative eigenvalues as there are negative entries in $\boldsymbol{D}$ $\square$

✍4 Assume that $\boldsymbol{A}$ is tridiagonal. How many operations are required to determine the number of negative eigenvalues of $\boldsymbol{A}$?

**Solution:** The rough answer is $\boldsymbol{O(n)}$ – because an LU (and therefore LDLT) factorization costs $\boldsymbol{O(n)}$. Based on doing the LU factorization of a triagonal matrix, a more accurate answer is $\boldsymbol{3n}$ operations. $\square$

✍5 Devise an algorithm based on the inertia theorem to compute the $\boldsymbol{i}$-th eigenvalue of a tridiagonal matrix.

**Solution:** Here is a matlab script:

```
   function [sigma] = bisect(d, b, i, tol)
%% function [sigma] = bisect(d, b, i, tol)
%% d   = diagonal of T
%% b   = co-diagonal
%% i   = compute i-th eigenvalue
%% tol = tolerance used for stopping
     b(1) = 0;
   n = length(d);
 %%-------------------- guershgorin
```

```
tmin = d(n) - abs(b(n));
tmax = d(n) + abs(b(n));
for j=1:n-1
  rho = abs(b(j)) + abs(b(j+1));
  tmin = min(tmin, d(j)-rho);
  tmax = max(tmax, d(j)+rho);
end
tol = tol*(tmax-tmin);
for iter=1:100
  sigma = 0.5*(tmin+tmax);
  count = sturm(d, b, sigma);
  if (count >= i)
    tmin = sigma;
  else
    tmax = sigma;
  end
  if (tmax - tmin) < tol
    break
  end
end
```

□

✍6 What is the inertia of the matrix

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \boldsymbol{F}^T & \boldsymbol{0} \end{pmatrix}$$

where $\boldsymbol{F}$ is $\boldsymbol{m} \times \boldsymbol{n}$, with $\boldsymbol{n} < \boldsymbol{m}$, and of full rank?

[Hint: use a block LU factorization]

**Solution:** We start with

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \boldsymbol{F}^T & 0 \end{pmatrix} = \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ 0 & -\boldsymbol{F}^T\boldsymbol{F} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{F}^T\boldsymbol{F} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ 0 & \boldsymbol{I} \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{F}^T\boldsymbol{F} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix}^T$$

This is of the form $\boldsymbol{XDX}^T$ where $\boldsymbol{X}$ is invertible.. Therefore the inertia is the same as that of the block diagonal matrix which is: $\boldsymbol{m}$ positive eigenvalues (block $\boldsymbol{I}$) and $\boldsymbol{n}$ negative eigenvalues since $-\boldsymbol{F}^T\boldsymbol{F}$ is $\boldsymbol{n} \times \boldsymbol{n}$ and negative definite.

✍7 Let $\|\boldsymbol{A_O}\|_{\boldsymbol{I}} = \mathbf{max}_{i \neq j} |\boldsymbol{a_{ij}}|$. Show that

$$\|\boldsymbol{A_O}\|_{\boldsymbol{F}} \leq \sqrt{\boldsymbol{n(n-1)}}\|\boldsymbol{A_O}\|_{\boldsymbol{I}}$$

**Solution:** This is straightforward:

$$\|A_O\|_F^2 = \sum_{i \neq j} |a_{ij}|^2 \leq n(n-1) \max_{i \neq j} |a_{ij}|^2 = n(n-1)\|A_O\|_I^2.$$

$\square$

✍8 Use this to show convergence in the case when largest entry is zeroed at each step.

**Solution:** If we call $B_k$ the matrix $A_O$ after each rotation then we have according to result in the previous page and using the previous exercise:

$$\begin{aligned}
\|B_{k+1}\|_F^2 &= \|B_k\|_F^2 - 2a_{pq}^2 \\
&= \|B_k\|_F^2 - 2\|B_k\|_I^2 \\
&\leq \|B_k\|_F^2 - \frac{2}{n(n+1)}\|B_k\|_F^2 \\
&= \left[1 - \frac{2}{n(n+1)}\right] \|B_k\|_F^2
\end{aligned}$$

which shows that the norm will be decreasing by factor less than a constant that is less than one - therefore it converges to zero. $\square$